ON THE POLYCONVULUTION WITH THE WEIGHT FUNCTION $\gamma(y) = \sin y$ OF HARTLEY INTEGRAL TRANFORMS H_1, H_2, H_1 AND INTEGRAL EQUATION

ĐA CHẬP VỚI HÀM TRỌNG $\gamma(y) = \sin y$ CỦA CÁC PHÉP BIẾN ĐỔI HARTLEY H_1, H_2, H_1 VÀ PHƯƠNG TRÌNH TÍCH PHÂN

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ASBSTRACT: *In this article, we introduce and investigate a new polyconvolution with the weight function $\gamma(y) = \sin y$ of Hartley integral transforms H_1, H_2, H_1. Several algebraic properties and appbications of this polyconvolution to solving a class of integral equation of Toeplitz plus hankel type and a class of systems of integral equations are presented.*

Keywords: *Integral equation, convolution, generalized convolution, polyconvolution, Hartley integral transforms.*

TÓM TẮT: *Trong bài báo này, chúng tôi giới thiệu và nghiên cứu một đa chấp mới với hàm trọng $\gamma(y) = \sin y$ của các phép biến đổi tích phân Hartley H_1, H_2, H_1. Một số tính chất đại số và ứng dụng của đa tích chập này để giải một loại phương trình tích phân của kiểu Toeplitz cộng với hankel và hệ phương trình tích phân được trình bày.*

Từ khóa: *Phương trình tích phân, tích chập, tích chập suy rộng, đa chập, phép biến đổi tích phân Hartley.*

1. INTRODUCTION

The notion of polyconvolution was first proposed by Kakichev in 1997 [7]. According to this definition, the polyconvolution for n+1 arbitrary integral transforms $K, K_1, K_2, ..., K_n$ with the weight function $\gamma(y)$ of functions f_1, f_2, ..., f_n satisfies the following factorization property $K\left[\overset{\gamma}{*}(f_1, f_2, \dots, f_n)\right](y) = \gamma(y)(K_1 f_1)(y)\dots)(K_n f_n)(y)$.

In recent time, there were some polyconvolution [11, 12] related to the Hartley integral transforms and some differential integral transforms. At the same time, there were some polyconvolutions [8, 9] only related to the Hartley integral transforms.

In this article, we expand to construct and investigate the new polyconvolution with the weight function $\gamma(y) = \sin y$ related to Hartley integral transforms H_1, H_2, H_1. We will apply this new polyconvolution to solving some no-standard integral equations and systems of integral equation. We realize that for such integral equations, a representation of their solutions in a closed form is interesting and open problem [6].

In this section, we recall some known convolutions, generalized convolutions.

The Hartley integral transforms H_1, H_2 were introduced in [3]

$(Hf)_{\{\substack{1\\2}\}}(x) =$
$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)cas(\pm xy)dy, x \in \mathbb{R}$

Here $\mathrm{cas}(\pm\theta) = \cos\theta \pm \sin\theta$.

The convolution for the Hartley integral [4, 5].

$\left(f\underset{H_1}{*}g\right)(x) = \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)[g(x+u) + g(x-u) + g(u-x) - g(-x-u)]du$ (1.1)

Satisfies the factorization property

$H_1(f\underset{H_1}{*}g)(y) = (H_1 f)(y) * (H_1 g)(y)$. (1.2)

The generalized convolution for the Hartley integral transform [2]

$\left(f\underset{H_1,H_1,H_2}{*}g\right)(x) = \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}[f(x+y) + f(x-y) - f(-x+y) + f(-x-y)]g(y)dy$, (1.3)

satisfies the factorization property

$H_1(f\underset{H_1,H_1,H_2}{*}g)(y) = (H_1 f)(y)(H_2 g)(y)$. (1.4)

2. POLYCONVOLUTION WITH THE WEIGHT FUNCTION $\gamma(y) = \sin y$ FOR HARTLEY INTEGRAL TRANSFORMS H_1, H_2, H_1

Definition 2.1. The polyconvolution with the weight function $\gamma(y) = \sin y$ for Hartley integral transforms H_1, H_2, H_1 of the functions f, g and h is defined as follow

$\left[\overset{\gamma}{*}(f,g,h)\right](x) = \frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[-f(-x-v-w-1) - f(-x+v+w-1) + f(-x+v+w+1 + f(-x-v-w+1) + f(x-v+w-1) - f(x+v-w-1) - f(x-v-w+1) + f(x+v-w+1)]g(v)h(w)dvdw.$

(2.1)

Theorem 2.2. Let f, g and $h \in L(R)$. Then the polyconvolution with the weight function $\gamma(y) = \sin y$ (2.1) for the Hartley integral transforms H_1, H_2, H_1 of the functions $f, g, h \in L(\mathbb{R})$. and the factorization identity holds

$H_1\left[\overset{\gamma}{*}(f,g,h)\right](y) =$
$\sin(y)\,(H_1 f)(y)(H_2 g)(y)(H_1 f), \forall y \in \mathbb{R}$ (2.2)

Proof. First of all, we prove that $\left[\overset{\gamma}{*}(f,g,h)\right](x) \in L(\mathbb{R})$ Indeed, we have estimation

$$\int_{-\infty}^{\infty} \gamma(f,g,h)|dx$$
$$\leq \frac{1}{8\pi}\int_{-\infty}^{\infty}|g(v)|dv \int_{-\infty}^{\infty}|h(w)|dw \int_{-\infty}^{\infty}[|f(-x-v-w-1)| + |f(-x+v+w-1)| + |f(-x+v+w+1| + |f(-x-v-w+1)| + |f(x-v+w-1)| + |f(x+v-w-1)| + |f(-x-v+w+1)| + |f(x+v-w+1)|]dx.$$

It is easy to see that

$$\int_{-\infty}^{\infty} \gamma(f,g,h)|dx$$
$$\leq \frac{1}{8\pi}\int_{-\infty}^{\infty}|g(v)|dv \int_{-\infty}^{\infty}|h(w)|dw \int_{-\infty}^{\infty}[|f(-x-v-w-1)| + |f(-x+v+w-1)| + |f(-x+v+w+1| + |f(-x-v-w+1)| + |f(x-v+w-1)| + |f(x+v-w-1)| + |f(-x-v+w+1)| + |f(x+v-w+1)|]dx = 8\int_{-\infty}^{\infty}|f(t)|\,dt.$$

For this reason, we obtain

$$\int_{-\infty}^{\infty} \overset{\gamma}{*}(f,g,h)(x)\Big|dx$$
$$\leq \frac{1}{\pi}\int_{-\infty}^{\infty}|g(v)|dv \int_{-\infty}^{\infty}|h(w)|dw \int_{-\infty}^{\infty}|f(t)|dt < +\infty.$$

So, $\left[\overset{\gamma}{*}(f,g,h)\right](x) \in L(\mathbb{R})$. Now we prove the factorization identity (2.2). Since

$$\sin y(H_1 f)(y)(H_2 g)(y)(H_1 h)(y) = \frac{1}{2\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin(y)\, cas(yu)cas(-yv)cas(yw)(fu)\, g(v)h(w)dudvdw$$

and using the trigonometric identity, we get

$$\sin y\, cas(yu)cas(-yv)cas(yw) = \frac{1}{4}[-casy(u-v-w-1) - casy(-u+v+w-1) + casy(-u+v+w+1 + casy(-u-v-w+1 + casy(u+v-w+1 - casy((u-v+w+1) - casy(u+v-w-1) + casy(-u+v+w+1]$$

That

$$\sin y(H_1 f)(y)(H_2 g)(y)(H_1 h)(y) = \frac{1}{8\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[-casy(u-v-w-1) - casy(-u+v+w-1) + casy(-u+v+w+1 + casy(-u-v-w+1 + casy(u+v-w+1 - casy((u-v+w+1) - casy(u+v-w-1) + casy(u-v+w-1]f(u)g(v)h(w)dudvdw$$

$$= \frac{1}{8\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -cas(yt)[-f(-t-v-w-1) - f(-t+v+w-1) + f(-t+v+w+1) + f(-t-v-w+1) + f(t-v+w+1) - f(t+v-w-1) - f(t-v+w+1) + f(t+v-w+1)]g(v)h(w)dtdvdw.$$

$$= H_1\left[\overset{\gamma}{*}(f,g,h)\right](y), \forall y \in \mathbb{R}.$$

Theorem 2.2 is proved.

Lemma 2.3. In the space $L(\mathbb{R})$ the polyconvolution (2.1) has following equality

$$[\overset{\gamma}{*}(f,g,h)](x) = [\overset{\gamma}{*}(h,g,f)](x).$$

Proof. Indeed, from Theorem 2.2, we get

$$H\left[\overset{\gamma}{*}(f,g,h)\right](y) = \sin(y)\,(H_1 f)(y)(H_2 g)(y)(H_1 h)(y) = \sin y\,(H_1 h)(y)(H_2 g)(y)(H_1 f)(y) = H\left[\overset{\gamma}{*}(f,g,h)\right](y), \forall y \in \mathbb{R}$$

implies that

$$[\overset{\gamma}{*}(f,g,h)](x) = [\overset{\gamma}{*}(h,g,f)](x).$$

In the sequel, we define the norm in the space $L(\mathbb{R})$ as follow

$$\|f\| = \frac{1}{\sqrt[3]{2\pi}}\int_{-\infty}^{\infty}|f(x)|dx.$$

Theorem 2.4. If f, g and h belong to $L(\mathbb{R})$, then the following inequality holds

$$\|\overset{\gamma}{*}(f,g,h)\| \le \|f\|.\|g\|.\|h\|.$$

Proof. From the proof of Theorem 2.2, we obtain:

$$\int_{-\infty}^{\infty}\left|{}_{*}^{\gamma}(f,g,h)(x)\right|dx$$
$$\leq\frac{1}{2\pi}\int_{-\infty}^{\infty}|f(t)|dt\int_{-\infty}^{\infty}|g(v)|dv\int_{-\infty}^{\infty}|h(w)|dw$$
$$\leq\frac{1}{\sqrt[3]{2\pi}}\int_{-\infty}^{\infty}|f(t)|dt\frac{1}{\sqrt[3]{2\pi}}\int_{-\infty}^{\infty}|g(v)|dv\frac{1}{\sqrt[3]{2\pi}}\int_{-\infty}^{\infty}|h(w)|dw.$$

So $|{}_{*}^{\gamma}(f,g,h)|\leq\|f\|.\|g\|.\|h\|$. The proof is completed.

Theorem 2.4. Let $g\in L_p(\mathbb{R})$, $h\in L_q(\mathbb{R})$, and $f\in L_r(\mathbb{R})$ such that $p,q,r>1$ and $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2$. Then the following inequality holds

$$\left|{}_{*}^{\gamma}(f,g,h)\right|\leq\frac{1}{2\pi}\|g\|_{L_p(\mathbb{R})}^{p}\|h\|_{L_p(\mathbb{R})}^{q}\|g\|_{L_p(\mathbb{R})}^{r}$$

Proof. From (2.1), we have the following estimation

$$\left|{}_{*}^{\gamma}(f,g,h)(x)\right|\leq$$
$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(v)||h(w)||f(-x-v-w-1)|dvdw+$$
$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(v)||h(w)||f(-x+v+w-1|dvdw+$$
$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(v)||h(w)||f(-x+v+w+1|dvdw+$$
$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(v)||h(w)||f(-x-v-w+1|dvdw+$$
$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(v)||h(w)||f(x-v+w-1|dvdw+$$
$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(v)||h(w)||f(x+v-w-1|dvdw+$$
$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(v)||h(w)||f(x-v+w+1|dvdw+$$
$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(v)||h(w)||f(x+v-w+1|dvdw. \quad \textbf{(2.3)}$$

Let $I_1,I_2,\cdots,I_7$ and I_8 be the corresponding integral terms in the above expression. Without loss of generality we consider

$$I_1=\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(v)|.|h(w)|.|f(-x-v-w-1)|dvdw, x\in\mathbb{R}.$$

Let p_1,q_1,r_1 be conjugate exponential of p,q,r and

$$B_1(u,v)=|h(w)|^{q/p_1}|f(-x-v-w-1)|^{r/p_1}\in L_{p_1}(\mathbb{R}^2)$$
$$B_2(u,v)=|f(-x-v-w-1)|^{\frac{r}{p_1}}|g(v)|^{q/p_1}\in L_{p_1}(\mathbb{R}^2)$$
$$B_3(u,v)=|g(w)|^{q/p_1}|h(w)|^{r/p_1})\in L_{p_1}(\mathbb{R}^2)$$

We see that

$$B_1.B_2.B_3=|g(v)|.|h(w)|.|f(-x-v-w-1)|$$

Using the definition of the norm on the space $L_{p1}(\mathbb{R}^2)$ along with the help of the Fubini theorem, we get

$$\|B_1\|_{L_{p_1}(R^2)}^{p_1}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left|h(w)^{q/p_1}\right||(-x-v-w-1|^{e/p_1}\right]^{p_1}dvdw$$
$$=\int_{-\infty}^{\infty}|h(w)^q|.\|f\|^r dw=\|h\|_{L_q(R)}^{q}.\|f\|_{L_r(R)}^{r}$$

Similarly, we have

$$\|B_2\|_{L_{q_1}(R^2)}^{q_1}=\|f\|_{L_r(R)}^{r}.\|g\|_{L_p(R)}^{p}$$

$$\|B_3\|^{r_1}_{L_{r_1}(R^2)} = \|fg\|^{p}_{L_p(R)}.\|h\|^{q}_{L_q(R)}$$
(2.4)

From the hypothesis $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2$, it follows that $\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1}=1$. Using Hölder inequality and (2.4), we have following estimation:

$$I_1 = \frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_1.B_2.B_3 dvdw$$

$$\leq \frac{1}{8\pi}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_1^{p_1} dvdw\right)^{\frac{1}{p_1}}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_2^{q_1} dvdw\right)^{\frac{1}{q_1}}$$

$$\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_3^{r_1} dvdw\right)^{\frac{1}{r_1}}=$$

$$\frac{1}{8\pi}\|B_1\|^{p_1}_{L_{p_1(p^2)}}.\|B_2\|^{q_1}_{L_{q_1(q^2)}}.\|B_3\|^{r_1}_{L_{r_1(r^2)}} =$$

$$\frac{1}{8\pi}\|g\|^{p_1}_{L_{p_1(p)}}.\|h\|^{q_1}_{L_{q_1(q)}}.\|f\|^{r_1}_{L_{r_1(r)}}$$

In the same way, we have the following estimations for $I_2, I_3, \cdots, I_7$, and I_8:

$$I_k \leq \frac{1}{8\pi}\|g\|^{p_1}_{L_{p_1(p)}}.\|h\|^{q_1}_{L_{q_1(q)}}.\|f\|^{r_1}_{L_{r_1(r)}}$$
2.5

for all $k = 2,3,\cdots,8$. Furthermore (2.3)-(2.5), we get

$$\left\|{}^{\gamma}_{*}(f,g,h)\right\| \leq \frac{1}{\pi}\|g\|^{p}_{L_{p(p)}}.\|h\|^{q}_{L_{q(q)}}.\|f\|^{r}_{L_{r(r)}}$$

Theorem 2.5. (Titchmarch - type Theorem)

Let $f,g,h \in L(\mathbb{R})$. If $\forall x \in \mathbb{R}, \left[{}^{\gamma}_{*}(f,g,h)\right](x) \equiv 0$, then either $f(x)=0$, or $g(x)=0$, or $h(x)=0, \forall x \in \mathbb{R}$.

Proof. The hypothesis $[{}^{\gamma}_{*}(f,g,h)](x) \equiv 0$, implies that $H\left[{}^{\gamma}_{*}(f,g,h)\right](y) = 0, \forall y \in \mathbb{R}$

Due to Theorem (2.2), we get

$$siny(H_1f)(y)(H_2g)(y)(H_1h)(y) = 0, \forall y \in \mathbb{R}$$

As $(H_1f)(y),(H_2g)(y),(H_1h)(y)$ are analytic $\forall y \in \mathbb{R}$ (2.6) implies that $\forall y \in \mathbb{R}$ $(H_1f) = 0, \forall y \in \mathbb{R}$ or $(H_2g) = 0, \forall y \in \mathbb{R}$ or $(H_1h) = 0, \forall y \in \mathbb{R}$. It follows that either $f(x) = 0, \forall x \in \mathbb{R}$, or $g(x) = 0, \forall x \in \mathbb{R}$, or $h(x) = 0, \forall x \in \mathbb{R}$.

3. APPLICATION TO SOLVING AN INTEGRAL EQUATION AND SYSTEM OF POLYCONVUTION TYPE

3.1 A single integral equation

In this subsection we apply the obtained result in solving an integral equation of polyconvolution type. To deal with this equation, we prove the existence of a solution as well as express it in closed form. We examine the following integral equation.1

$$f(x) + \frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[-f(-x-v-w-1) - f(-x+v+w-1) + f(-x+v+w+1 + f(-x-v-w+1) + f(x-v+w-1) - f(x+v-w-1) - f(x-v-w+1) + f(x+v-w+1)]g(v)h(w)dvdw = k(x), x \in \mathbb{R} \quad (3.1)$$

Here g, f, and k are functions of $L(\mathbb{R})$, f is an unknown function.

Theorem 3.1. Let $k,g,h \in L(\mathbb{R})$. be given. Equation (3.1) has a unique solution in $L(\mathbb{R})$ if $1 + siny(H_2g)(y)(H_1h)(y) \neq 0, \forall y \in \mathbb{R}$. The solution of f(x) is defined by: $f(x) = k(x) - (k^{*}_{H_1}\ l)(x)$. Here $l \in L(\mathbb{R})$ and it is determined by:

$$(H_1l)(y) = \frac{\sin y(H_2g)(y)(H_1h)(y)}{1+\sin y(H_2g)(y)(H_1h)(y)}.$$

The equation (3.1) can be rewritten in the form

$$f(x)+[\overset{\gamma}{*}(f,g,h)](x)=k(x)$$

Due to Theorem (2.2), we have

$$(H_1f)(y) + sin(H_1f)(y)(H_2g)(y)(H_1h)(y) = (H_1k)(y), y \in \mathbb{R}$$

$$(H_1f)(y) + [1+siny(H_2g)(y)(H_1h)(y)] = (H_1k)(y), y \in \mathbb{R}$$

With the condition $1 + siny(H_2g)(y)(H_1h)(y) \neq 0, y \in \mathbb{R}$, we get

$$(H_1f)(y)=(H_1k)(y)\left[1-\frac{\sin y(H_2g)(y)(H_1h)(y)}{1+\sin y(H_2g)(y)(H_1h)(y)}\right].$$

Therefore, by Wierner-Levy's theorem ([1, 8]), there exists a function $l \in L(\mathbb{R})$ such that

$$(H_1l)(y)=\frac{\sin y(H_2g)(y)(H_1h)(y)}{1+\sin y(H_2g)(y)(H_1h)(y)}.$$

Hence

$$(H_1f)(y) = (H_1k)(y) - [1-(Hl)(y)] = (H_1k)(y) - H_1(k\overset{*}{_{H_1}} l)(x) \in L(\mathbb{R}).$$

Thus, $f(x) = k(x) - (k\overset{*}{_{H_1}} l)(x) \in L(\mathbb{R})$.

3.2 A system of two integral equations of polyconvolution type

Now we consider the following system

$$f(x)+\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[-g(-x-v-w-1)-g(-x+v+w-1)+g(-x+v+w+1+g(-x-v-w+1)+g(x-v+w-1)-g(x+v-w-1)-g(x-v+w+1)+g(x+v-w+1)]\varphi(v)\varphi(w)dvdw = h(x)$$
$$\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}f(v)[p(x+v)+p(x-v)+p(-x+v)-p(-x-v)]d(v)+g(v) = k(x), x \in \mathbb{R}(3.2)$$

Here φ, ψ, p, h, and k are given functions in $L(\mathbb{R})$, f and g are the unknown functions.

Theorem 3.2. Under the condition $1 - H_1[\overset{\gamma}{*}p,\varphi,\psi](y) \neq 0, \forall y \in \mathbb{R}$, there exists a unique solution in $L(\mathbb{R})$ of (3.2) which is defined by

$$\begin{cases} f(x)=h(x)+(h\underset{H_1}{*}l)(x)-[\overset{\gamma}{*}(k,\varphi,\psi)](x)-[\overset{\gamma}{*}(k,\varphi,\psi)]^{*}_{H_1}l](x) \\ g(x)=k(x)+(k\underset{H_1}{*}l)(x)-(h\underset{H_1}{*}p)(x)-[(h\underset{H_1}{*}p)\underset{H_1}{*}l](x). \end{cases}$$

Here $l \in L(\mathbb{R})$ and defined by

$$(\mathcal{H}_1l)(y)=\frac{H_1[\overset{\gamma}{*}(p,\varphi,\psi)](y)}{1-H_1[\overset{\gamma}{*}(p,\varphi,\psi)](y)}.$$

Proof. System (3.2) can be written in the form

$$\begin{cases} f(x)+[\overset{\gamma}{*}(g,\varphi,\psi](x)=h(x) \\ (f\overset{*}{_{H_1}}p)(x)+g(x)=k(x), x\in\mathbb{R} \end{cases}$$

Using the factorization property of the polyconvolution (1.1) and the convolution (2.1) we get the linear system of algebraic equations with respectively to $(H_1f)(y)$ and $(H_1g)(y)$:

$$\begin{cases} (H_1f)(y)+siny(H_1g)(y).(H_2\varphi)(y)(H_1\psi)(y)=(H_1h)(y) \\ (H_1f)(y)(H_1p)(y)+(H_1g)(y)=(H_1k)(y), y\in\mathbb{R} \end{cases}$$

We calculate the determinants of the system

$$\Delta=\begin{vmatrix} 1 & \sin y(H_2\varphi)(y)(H_1\psi)(y) \\ (H_1p)(y) & 1 \end{vmatrix} = 1-\sin y(H_1p)(y)(H_2\varphi)(y)(H_1\psi)(y) = 1-H_1[\overset{\gamma}{*}(p,\varphi,\psi)](y),$$

$$\Delta_1=\begin{vmatrix} (H_1h)(y) & \mathrm{siny}(H_2\varphi)(y)(H_1\psi)(y) \\ (H_1k)(y) & 1 \end{vmatrix} = (H_1h)(y) - H_1[\overset{\gamma}{*}(k,\varphi,\psi)](y),$$

$$\Delta_2 = \begin{vmatrix} 1 & (H_1h)(y) \\ (H_1p)(y) & (H_1k)(y) \end{vmatrix} = (H_1k)(y) - H_1\left(h^*_{H_1}p\right)(y).$$

Since $1 - H_1\left[{}^{\gamma}_{*}(p,\varphi,\psi)\right](y) \neq 0, \forall y \in \mathbb{R}$, we have

$$(H_1f)(y) = \left\{(H_1h)(y) - H_1[{}^{\gamma}_{*}(k,\varphi,\psi)](y)\right\}\frac{1}{1-H_1[{}^{\gamma}_{*}(p,\varphi,\psi)](y)}$$

$$= \left\{(H_1h)(y) - H_1[{}^{\gamma}_{*}(k,\varphi,\psi)](y)\right\}\left\{1+\frac{H_1[{}^{\gamma}_{*}(p,\varphi,\psi)](y)}{1-H_1[{}^{\gamma}_{*}(p,\varphi,\psi)](y)}\right\}$$

Furthermore, according to Wiener-Levy's theorem ([1, 10]), there exists a function $l \in L(\mathbb{R})$ such that

$$(H_1l)(y) = \frac{H_1[{}^{\gamma}_{*}(p,\varphi,\psi)](y)}{1-H_1[{}^{\gamma}_{*}(p,\varphi,\psi)](y)}.$$

It follows that

$$(H_1f)(y) = \left\{(H_1h)(y) - H_1\left[{}^{\gamma}_{*}(k,\varphi,\psi)\right](y)\right\}.\{1 + (H_1l)(y)\} = (H_1h)(y) + H_1\left(h^*_{H_1}l\right)(y) - H_1[{}^{\gamma}_{*}(k,\varphi,\psi)](y) - H_1\{[{}^{\gamma}_{*}(k,\varphi,\psi)]{}^{\gamma}_{*}H_1l\}(y).$$

So,

$$f(x) = h(x) + \left(h^*_{H_1}l\right)(x) - \left[{}^{\gamma}_{*}(k,\varphi,\psi)\right](x) - \left\{\left[{}^{\gamma}_{*}(k,\varphi,\psi)\right]{}^{\gamma}_{*}H_1l\right\}(x) \in L(\mathbb{R}).$$

In the same way, we get

$$\left(H_g\right)(y) = \{(H_1k)(y) - H_1(h^*_{H_1}p)(y)\}.\{1 + (H_1l)(y)\} = (H_1k)(y) - H_1(k^*_{H_1}l)(y) - H_1\left[\left(h^*_{H_1}p\right){}_{H_1}{}^{*}l\right](y).$$

It follows that

$$g(x) = k(x) + \left(k^*_{H_1}l\right)(x) - \left(h^*_{H_1}p\right)(x) - \left[\left(h^*_{H_1}p\right){}_{H_1}{}^{*}l\right](x) \in L(\mathbb{R}).$$

4. CONCLUSIONS

In this paper, we presented the new polyconvolution with the weight function of three functions $\gamma(y) = \sin y$ of Hartley integral transforms H_1, H_2, H_1 and proved several algebraic properties of the polyconvolution. Some norm inequalities for this polyconvolution operator in the function spaces and are established. As a result, a class of Toeplitz plus Hankel integral equation and a system of integral equations can be solved in a closed form.

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