JAMES STEWART

ESSENTIAL CALCULUS

EARLY TRANSCENDENTALS

∫

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ALGEBRA

ARITHMETIC OPERATIONS

$$a(b+c) = ab + ac \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

$$\frac{a+c}{b} = \frac{a}{b} + \frac{c}{b} \qquad \frac{\dfrac{a}{b}}{\dfrac{c}{d}} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

EXPONENTS AND RADICALS

$$x^m x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n}$$

$$(x^m)^n = x^{mn} \qquad x^{-n} = \frac{1}{x^n}$$

$$(xy)^n = x^n y^n \qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^{1/n} = \sqrt[n]{x} \qquad x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$$

$$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y} \qquad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

FACTORING SPECIAL POLYNOMIALS

$$x^2 - y^2 = (x+y)(x-y)$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

BINOMIAL THEOREM

$$(x+y)^2 = x^2 + 2xy + y^2 \qquad (x-y)^2 = x^2 - 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + nxy^{n-1} + y^n$$

where $\displaystyle\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot k}$

QUADRATIC FORMULA

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

INEQUALITIES AND ABSOLUTE VALUE

If $a < b$ and $b < c$, then $a < c$.

If $a < b$, then $a + c < b + c$.

If $a < b$ and $c > 0$, then $ca < cb$.

If $a < b$ and $c < 0$, then $ca > cb$.

If $a > 0$, then

$|x| = a$ means $x = a$ or $x = -a$

$|x| < a$ means $-a < x < a$

$|x| > a$ means $x > a$ or $x < -a$

GEOMETRY

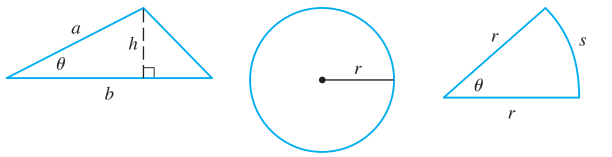
GEOMETRIC FORMULAS

Formulas for area A, circumference C, and volume V:

Triangle
$A = \frac{1}{2}bh$
$= \frac{1}{2}ab\sin\theta$

Circle
$A = \pi r^2$
$C = 2\pi r$

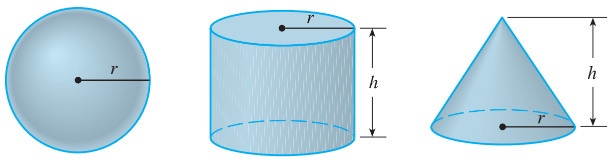
Sector of Circle
$A = \frac{1}{2}r^2\theta$
$s = r\theta$ (θ in radians)



Sphere
$V = \frac{4}{3}\pi r^3$
$A = 4\pi r^2$

Cylinder
$V = \pi r^2 h$

Cone
$V = \frac{1}{3}\pi r^2 h$



DISTANCE AND MIDPOINT FORMULAS

Distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of $\overline{P_1P_2}$: $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

LINES

Slope of line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-slope equation of line through $P_1(x_1, y_1)$ with slope m:

$$y - y_1 = m(x - x_1)$$

Slope-intercept equation of line with slope m and y-intercept b:

$$y = mx + b$$

CIRCLES

Equation of the circle with center (h, k) and radius r:

$$(x-h)^2 + (y-k)^2 = r^2$$

TRIGONOMETRY

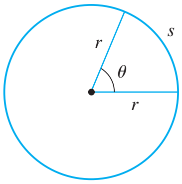
ANGLE MEASUREMENT

π radians $= 180°$

$1° = \dfrac{\pi}{180}$ rad $\qquad$ 1 rad $= \dfrac{180°}{\pi}$

$s = r\theta$

(θ in radians)

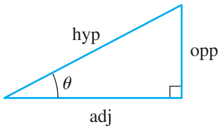


RIGHT ANGLE TRIGONOMETRY

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \dfrac{\text{hyp}}{\text{opp}}$

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \dfrac{\text{hyp}}{\text{adj}}$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}} \qquad \cot\theta = \dfrac{\text{adj}}{\text{opp}}$

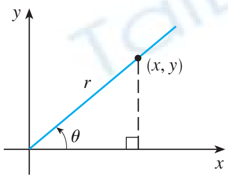


TRIGONOMETRIC FUNCTIONS

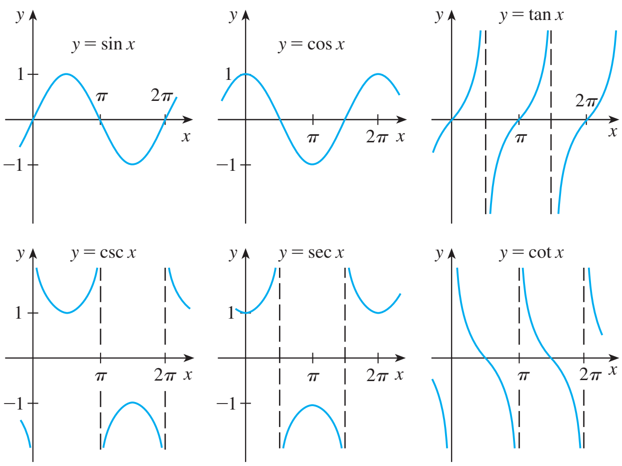
$\sin\theta = \dfrac{y}{r} \qquad \csc\theta = \dfrac{r}{y}$

$\cos\theta = \dfrac{x}{r} \qquad \sec\theta = \dfrac{r}{x}$

$\tan\theta = \dfrac{y}{x} \qquad \cot\theta = \dfrac{x}{y}$



GRAPHS OF THE TRIGONOMETRIC FUNCTIONS



TRIGONOMETRIC FUNCTIONS OF IMPORTANT ANGLES

θ	radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	0	1	0
$30°$	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$45°$	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$60°$	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$90°$	$\pi/2$	1	0	—

FUNDAMENTAL IDENTITIES

$\csc\theta = \dfrac{1}{\sin\theta} \qquad \sec\theta = \dfrac{1}{\cos\theta}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} \qquad \cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\cot\theta = \dfrac{1}{\tan\theta} \qquad \sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$

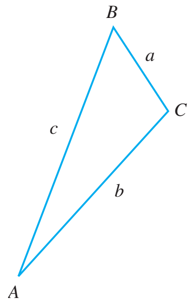
$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta \qquad \sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

$\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta \qquad \tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta$

THE LAW OF SINES

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



THE LAW OF COSINES

$a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

ADDITION AND SUBTRACTION FORMULAS

$\sin(x + y) = \sin x\cos y + \cos x\sin y$

$\sin(x - y) = \sin x\cos y - \cos x\sin y$

$\cos(x + y) = \cos x\cos y - \sin x\sin y$

$\cos(x - y) = \cos x\cos y + \sin x\sin y$

$\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y}$

$\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x\tan y}$

DOUBLE-ANGLE FORMULAS

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

HALF-ANGLE FORMULAS

$\sin^2 x = \dfrac{1 - \cos 2x}{2} \qquad \cos^2 x = \dfrac{1 + \cos 2x}{2}$

ESSENTIAL CALCULUS
EARLY TRANSCENDENTALS

JAMES STEWART
McMaster University and University of Toronto

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PREFACE

This book is a response to those instructors who feel that calculus textbooks are too big. In writing the book I asked myself: What is essential for a three-semester calculus course for scientists and engineers?

The book is about two-thirds the size of my other calculus books (*Calculus,* Fifth Edition and *Calculus, Early Transcendentals,* Fifth Edition) and yet it contains almost all of the same topics. I have achieved relative brevity mainly by condensing the exposition and by putting some of the features on the website **www.stewartcalculus.com**. Here, in more detail are some of the ways I have reduced the bulk:

- I have organized topics in an efficient way and rewritten some sections with briefer exposition.
- The design saves space. In particular, chapter opening spreads and photographs have been eliminated.
- The number of examples is slightly reduced. Additional examples are provided online.
- The number of exercises is somewhat reduced, though most instructors will find that there are plenty. In addition, instructors have access to the archived problems on the website.
- Although I think projects can be a very valuable experience for students, I have removed them from the book and placed them on the website.
- A discussion of the principles of problem solving and a collection of challenging problems for each chapter have been moved to the web.

Despite the reduced size of the book, there is still a modern flavor: Conceptual understanding and technology are not neglected, though they are not as prominent as in my other books.

CONTENT

This book treats the exponential, logarithmic, and inverse trigonometric functions early, in Chapter 3. Those who wish to cover such functions later, with the logarithm defined as an integral, should look at my book titled simply *Essential Calculus.*

CHAPTER 1 ▪ FUNCTIONS AND LIMITS After a brief review of the basic functions, limits and continuity are introduced, including limits of trigonometric functions, limits involving infinity, and precise definitions.

CHAPTER 2 ▪ DERIVATIVES The material on derivatives is covered in two sections in order to give students time to get used to the idea of a derivative as a function. The formulas for the derivatives of the sine and cosine functions are derived in the section on basic differentiation formulas. Exercises explore the meanings of derivatives in various contexts.

CHAPTER 3 ▪ INVERSE FUNCTIONS: EXPONENTIAL, LOGARITHMIC, AND INVERSE TRIGONOMETRIC FUNCTIONS Exponential functions are defined first and the number e is defined as a limit. Logarithms are then defined as inverse functions. Applications to exponential growth and decay follow. Inverse trigonometric functions and hyperbolic functions are also covered here. L'Hospital's Rule is included in this chapter because limits of transcendental functions so often require it.

CHAPTER 4 ▪ APPLICATIONS OF DIFFERENTIATION The basic facts concerning extreme values and shapes of curves are deduced from the Mean Value Theorem. The section on curve sketching includes a brief treatment of graphing with technology. The section on optimization problems contains a brief discussion of applications to business and economics.

CHAPTER 5 ▪ INTEGRALS The area problem and the distance problem serve to motivate the definite integral, with sigma notation introduced as needed. (Full coverage of sigma notation is provided in Appendix C.) A quite general definition of the definite integral (with unequal subintervals) is given initially before regular partitions are employed. Emphasis is placed on explaining the meanings of integrals in various contexts and on estimating their values from graphs and tables.

CHAPTER 6 ▪ TECHNIQUES OF INTEGRATION All the standard methods are covered, as well as computer algebra systems, numerical methods, and improper integrals.

CHAPTER 7 ▪ APPLICATIONS OF INTEGRATION General methods are emphasized. The goal is for students to be able to divide a quantity into small pieces, estimate with Riemann sums, and recognize the limit as an integral. The chapter concludes with an introduction to differential equations, including separable equations and direction fields.

CHAPTER 8 ▪ SERIES The convergence tests have intuitive justifications as well as formal proofs. The emphasis is on Taylor series and polynomials and their applications to physics. Error estimates include those based on Taylor's Formula (with Lagrange's form of the remainder term) and those from graphing devices.

CHAPTER 9 ▪ PARAMETRIC EQUATIONS AND POLAR COORDINATES This chapter introduces parametric and polar curves and applies the methods of calculus to them. A brief treatment of conic sections in polar coordinates prepares the way for Kepler's Laws in Chapter 10.

CHAPTER 10 ▪ VECTORS AND THE GEOMETRY OF SPACE In addition to the material on vectors, dot and cross products, lines, planes, and surfaces, this chapter covers vector-valued functions, length and curvature of space curves, and velocity and acceleration along space curves, culminating in Kepler's laws.

CHAPTER 11 ▪ PARTIAL DERIVATIVES In view of the fact that many students have difficulty forming mental pictures of the concepts of this chapter, I've placed a special emphasis on graphics to elucidate such ideas as graphs, contour maps, directional derivatives, gradients, and Lagrange multipliers.

CHAPTER 12 ▪ MULTIPLE INTEGRALS Cylindrical and spherical coordinates are introduced in the context of evaluating triple integrals.

CHAPTER 13 ▪ VECTOR CALCULUS The similarities among the Fundamental Theorem for line integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem are emphasized.

WEBSITE

The website www.stewartcalculus.com includes the following.

- Review of Algebra, Analytic Geometry, and Conic Sections
- Additional Examples
- Projects
- Archived Problems (drill exercises that have appeared in previous editions of my other books), together with their solutions
- Challenge Problems
- Complex Numbers
- Graphing Calculators and Computers
- Lies My Calculator and Computer Told Me
- Additional Topics (complete with exercise sets): Principles of Problem Solving, Strategy for Integration, Strategy for Testing Series, Fourier Series, Area of a Surface of Revolution, Linear Differential Equations, Second-Order Linear Differential Equations, Nonhomogeneous Linear Equations, Applications of Second-Order Differential Equations, Using Series to Solve Differential Equations, Complex Numbers, Rotation of Axes
- Links, for particular topics, to outside web resources
- History of Mathematics, with links to the better historical websites

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The idea for this book came from my editor, Bob Pirtle, who had been hearing of the desire for a much shorter calculus text from numerous instructors. I thank him for encouraging me to pursue this idea and for his advice and assistance whenever I needed it.

JAMES STEWART

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TO THE STUDENT

Reading a calculus textbook is different from reading a newspaper or a novel, or even a physics book. Don't be discouraged if you have to read a passage more than once in order to understand it. You should have pencil and paper and calculator at hand to sketch a diagram or make a calculation.

Some students start by trying their homework problems and read the text only if they get stuck on an exercise. I suggest that a far better plan is to read and understand a section of the text before attempting the exercises. In particular, you should look at the definitions to see the exact meanings of the terms. And before you read each example, I suggest that you cover up the solution and try solving the problem yourself. You'll get a lot more from looking at the solution if you do so.

Part of the aim of this course is to train you to think logically. Learn to write the solutions of the exercises in a connected, step-by-step fashion with explanatory sentences—not just a string of disconnected equations or formulas.

The answers to the odd-numbered exercises appear at the back of the book, in Appendix E. Some exercises ask for a verbal explanation or interpretation or description. In such cases there is no single correct way of expressing the answer, so don't worry that you haven't found the definitive answer. In addition, there are often several different forms in which to express a numerical or algebraic answer, so if your answer differs from mine, don't immediately assume you're wrong. For example, if the answer given in the back of the book is $\sqrt{2} - 1$ and you obtain $1/(1 + \sqrt{2})$, then you're right and rationalizing the denominator will show that the answers are equivalent.

The icon indicates an exercise that definitely requires the use of either a graphing calculator or a computer with graphing software. But that doesn't mean that graphing devices can't be used to check your work on the other exercises as well. The symbol CAS is reserved for problems in which the full resources of a computer algebra system (like Derive, Maple, Mathematica, or the TI-89/92) are required. You will also encounter the symbol ⊘, which warns you against committing an error. I have placed this symbol in the margin in situations where I have observed that a large proportion of my students tend to make the same mistake.

The CD-ROM *Tools for Enriching™ Calculus* is referred to by means of the symbol TEC. It directs you to *Visuals* and *Modules* in which you can explore aspects of calculus for which the computer is particularly useful. TEC also provides *Homework Hints* for representative exercises that are indicated by printing the exercise number in blue: **43.**. These homework hints ask you questions that allow you to make progress toward a solution without actually giving you the answer. You need to pursue each hint in an active manner with pencil and paper to work out the details. If a particular hint doesn't enable you to solve the problem, you can click to reveal the next hint. (See the front endsheet for information on how to purchase this and other useful tools.)

The *Interactive Video Skillbuilder CD-ROM* contains videos of instructors explaining two or three of the examples in every section of the text. (The symbol V has been placed beside these examples in the text.) Also on the CD is a video in which I offer advice on how to succeed in your calculus course.

I also want to draw your attention to the website **www.stewartcalculus.com**. There you will find an *Algebra Review* (in case your precalculus skills are weak) as well as *Additional Examples, Challenging Problems, Projects, Lies My Calculator and Computer Told Me* (explaining why calculators sometimes give the wrong answer), *History of Mathematics, Additional Topics,* chapter quizzes, and links to outside resources.

I recommend that you keep this book for reference purposes after you finish the course. Because you will likely forget some of the specific details of calculus, the book will serve as a useful reminder when you need to use calculus in subsequent courses. And, because this book contains more material than can be covered in any one course, it can also serve as a valuable resource for a working scientist or engineer.

Calculus is an exciting subject, justly considered to be one of the greatest achievements of the human intellect. I hope you will discover that it is not only useful but also intrinsically beautiful.

JAMES STEWART

1 FUNCTIONS AND LIMITS

Calculus is fundamentally different from the mathematics that you have studied previously. Calculus is less static and more dynamic. It is concerned with change and motion; it deals with quantities that approach other quantities. So in this first chapter we begin our study of calculus by investigating how the values of functions change and approach limits.

1.1 FUNCTIONS AND THEIR REPRESENTATONS

Functions arise whenever one quantity depends on another. Consider the following four situations.

A. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a *function* of r.

B. The human population of the world P depends on the time t. The table gives estimates of the world population $P(t)$ at time t, for certain years. For instance,

$$P(1950) \approx 2{,}560{,}000{,}000$$

But for each value of the time t there is a corresponding value of P, and we say that P is a function of t.

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

C. The cost C of mailing a first-class letter depends on the weight w of the letter. Although there is no simple formula that connects w and C, the post office has a rule for determining C when w is known.

D. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t. Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t, the graph provides a corresponding value of a.

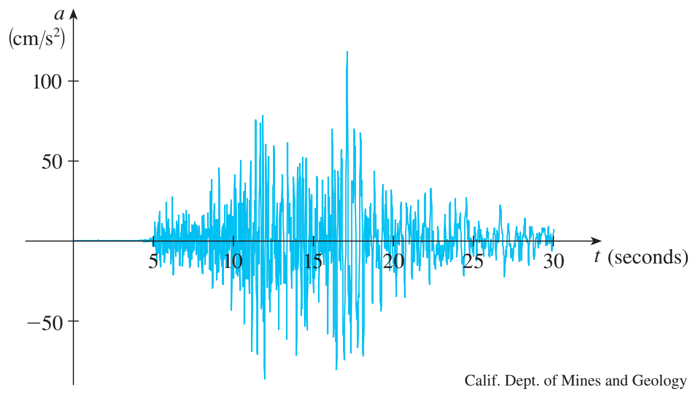

FIGURE 1
Vertical ground acceleration during the Northridge earthquake

Each of these examples describes a rule whereby, given a number (r, t, w, or t), another number (A, P, C, or a) is assigned. In each case we say that the second number is a function of the first number.

> A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

We usually consider functions for which the sets A and B are sets of real numbers. The set A is called the **domain** of the function. The number $f(x)$ is the **value of f at x** and is read "f of x." The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain. A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**. A symbol that represents a number in the *range* of f is called a **dependent variable**. In Example A, for instance, r is the independent variable and A is the dependent variable.

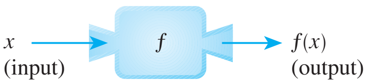

FIGURE 2
Machine diagram for a function f

It's helpful to think of a function as a **machine** (see Figure 2). If x is in the domain of the function f, then when x enters the machine, it's accepted as an input and the machine produces an output $f(x)$ according to the rule of the function. Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

Another way to picture a function is by an **arrow diagram** as in Figure 3. Each arrow connects an element of A to an element of B. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on.

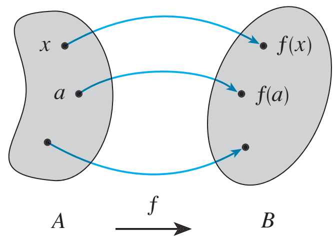

FIGURE 3
Arrow diagram for f

The most common method for visualizing a function is its graph. If f is a function with domain A, then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in A\}$$

(Notice that these are input-output pairs.) In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

The graph of a function f gives us a useful picture of the behavior or "life history" of a function. Since the y-coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x. (See Figure 4.) The graph of f also allows us to picture the domain of f on the x-axis and its range on the y-axis as in Figure 5.

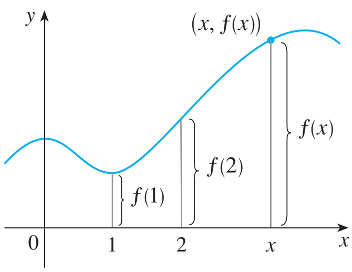

FIGURE 4

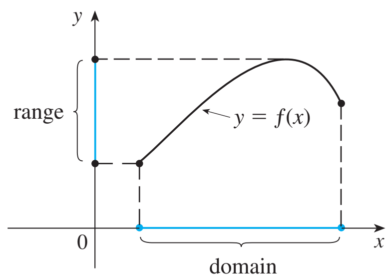

FIGURE 5

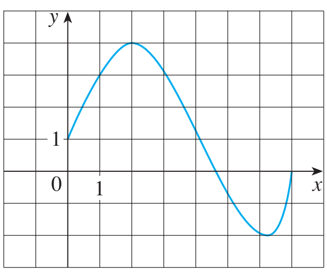

FIGURE 6

EXAMPLE 1 The graph of a function f is shown in Figure 6.
(a) Find the values of $f(1)$ and $f(5)$.
(b) What are the domain and range of f?

SOLUTION
(a) We see from Figure 6 that the point $(1, 3)$ lies on the graph of f, so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the x-axis.)

When $x = 5$, the graph lies about 0.7 unit below the x-axis, so we estimate that $f(5) \approx -0.7$.

▪ The notation for intervals is given on Reference Page 3. The Reference Pages are located at the front and back of the book.

(b) We see that $f(x)$ is defined when $0 \leqslant x \leqslant 7$, so the domain of f is the closed interval $[0, 7]$. Notice that f takes on all values from -2 to 4, so the range of f is

$$\{y \mid -2 \leqslant y \leqslant 4\} = [-2, 4]$$ ▪

REPRESENTATIONS OF FUNCTIONS

There are four possible ways to represent a function:

- verbally (by a description in words)
- visually (by a graph)
- numerically (by a table of values)
- algebraically (by an explicit formula)

If a single function can be represented in all four ways, it is often useful to go from one representation to another to gain additional insight into the function. But certain functions are described more naturally by one method than by another. With this in mind, let's reexamine the four situations that we considered at the beginning of this section.

A. The most useful representation of the area of a circle as a function of its radius is probably the algebraic formula $A(r) = \pi r^2$, though it is possible to compile a table of values or to sketch a graph (half a parabola). Because a circle has to have a positive radius, the domain is $\{r \mid r > 0\} = (0, \infty)$, and the range is also $(0, \infty)$.

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

B. We are given a description of the function in words: $P(t)$ is the human population of the world at time t. The table of values of world population provides a convenient representation of this function. If we plot these values, we get the graph (called a *scatter plot*) in Figure 7. It too is a useful representation; the graph allows us to absorb all the data at once. What about a formula? Of course, it's impossible to devise an explicit formula that gives the exact human population $P(t)$ at any time t. But it is possible to find an expression for a function that *approximates* $P(t)$. In fact, we could use a graphing calculator with exponential regression capabilities to obtain the approximation

$$P(t) \approx f(t) = (0.008079266) \cdot (1.013731)^t$$

and Figure 8 shows that it is a reasonably good "fit." The function f is called a *mathematical model* for population growth. In other words, it is a function with an explicit formula that approximates the behavior of our given function. We will see, however, that the ideas of calculus can be applied to a table of values; an explicit formula is not necessary.

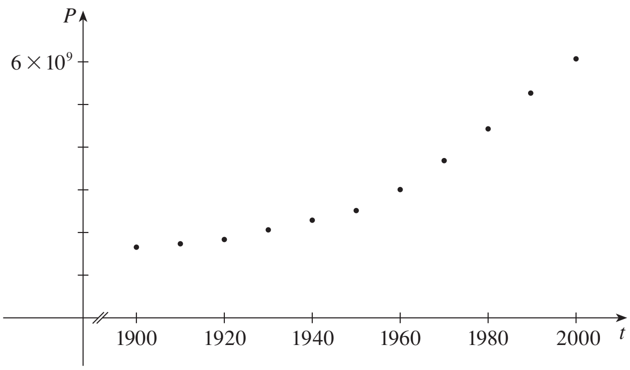

FIGURE 7 Scatter plot of data points for population growth

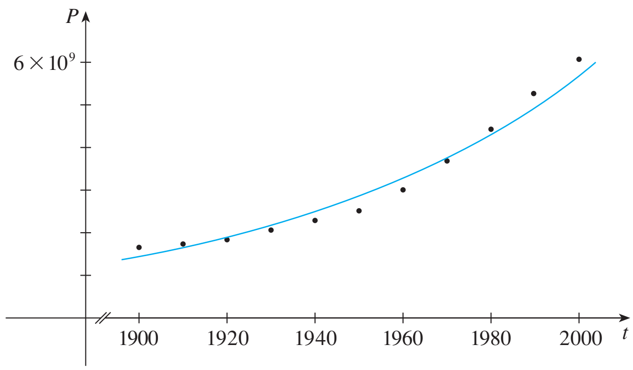

FIGURE 8 Graph of a mathematical model for population growth

▪ A function defined by a table of values is called a *tabular* function.

w (ounces)	$C(w)$ (dollars)
$0 < w \leqslant 1$	0.39
$1 < w \leqslant 2$	0.63
$2 < w \leqslant 3$	0.87
$3 < w \leqslant 4$	1.11
$4 < w \leqslant 5$	1.35
⋮	⋮
$12 < w \leqslant 13$	3.27

The function P is typical of the functions that arise whenever we attempt to apply calculus to the real world. We start with a verbal description of a function. Then we may be able to construct a table of values of the function, perhaps from instrument readings in a scientific experiment. Even though we don't have complete knowledge of the values of the function, we will see throughout the book that it is still possible to perform the operations of calculus on such a function.

C. Again the function is described in words: $C(w)$ is the cost of mailing a first-class letter with weight w. The rule that the US Postal Service used as of 2006 is as follows: The cost is 39 cents for up to one ounce, plus 24 cents for each successive ounce up to 13 ounces. The table of values shown in the margin is the most convenient representation for this function, though it is possible to sketch a graph (see Example 6).

D. The graph shown in Figure 1 is the most natural representation of the vertical acceleration function $a(t)$. It's true that a table of values could be compiled, and it is even possible to devise an approximate formula. But everything a geologist needs to know—amplitudes and patterns—can be seen easily from the graph. (The same is true for the patterns seen in electrocardiograms of heart patients and polygraphs for lie-detection.)

In the next example we sketch the graph of a function that is defined verbally.

EXAMPLE 2 When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

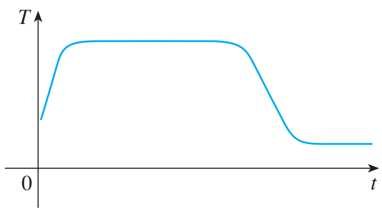

FIGURE 9

SOLUTION The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes. When the water from the hot-water tank starts flowing from the faucet, T increases quickly. In the next phase, T is constant at the temperature of the heated water in the tank. When the tank is drained, T decreases to the temperature of the water supply. This enables us to make the rough sketch of T as a function of t in Figure 9. ■

EXAMPLE 3 Find the domain of each function.

(a) $f(x) = \sqrt{x + 2}$ (b) $g(x) = \dfrac{1}{x^2 - x}$

SOLUTION

▪ If a function is given by a formula and the domain is not stated explicitly, the convention is that the domain is the set of all numbers for which the formula makes sense and defines a real number.

(a) Because the square root of a negative number is not defined (as a real number), the domain of f consists of all values of x such that $x + 2 \geqslant 0$. This is equivalent to $x \geqslant -2$, so the domain is the interval $[-2, \infty)$.

(b) Since

$$g(x) = \frac{1}{x^2 - x} = \frac{1}{x(x - 1)}$$

and division by 0 is not allowed, we see that $g(x)$ is not defined when $x = 0$ or $x = 1$. Thus the domain of g is $\{x \mid x \neq 0, x \neq 1\}$, which could also be written in interval notation as $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$. ■

The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions? This is answered by the following test.

THE VERTICAL LINE TEST A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.