EXISTENCE OF SOLUTIONS FOR A CLASS OF NONLOCAL AND NON-HOMOGENEOUS EQUATIONS IN ORLICZ-SOBOLEV SPACE

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Abstract*. In this paper, we investigate the existence of multiple solutions for a class of non-homogeneous Kirchhoff type problems in Orlicz-Sobolev spaces. Our results are established by using the mountain pass theorem combined with the Ekeland variational principle.*

Keywords: *Non-homogeneous operator; Orlicz-Sobolev spaces; Kirchhoff type problems; Variational methods*

1. INTRODUCTION

Let Ω be an open bounded subset of $\mathbb{R}^N$ $(N \geq 3)$, with smooth boundary $\partial\Omega$, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative. Assume that a: $\overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is such that the mapping φ: $\overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(x,t) = \begin{cases} a(x,|t|)t, & \text{for } t \neq 0, \\ 0, & \text{for } t=0, \end{cases}$$

satisfies the condition $\mathrm{H}(\varphi)$: for all $x \in \overline{\Omega}$, $\varphi(x,.)$ is an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

In this work, we deal with the following Kirchhoff type problems with Neumann boundary condition

$$\begin{cases} -M(L(u))\left[div\left(a\left(x,|\nabla u|\right)\nabla u\right) - a\left(x,|u|\right)u\right] = f(x,u) + g(x) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where f: $\overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, g: $\overline{\Omega} \to \mathbb{R}$ is a perturbation term and $M(t)$: $\mathbb{R}^+ = [0, +\propto) \to \mathbb{R}^+$ is a nondecreasing continuous function, and the functional L defined by

$$L(u) := \int_\Omega \left(\Phi\left(x,|\nabla u|\right) + \Phi\left(x,|u|\right)\right)dx, \tag{2}$$

where

$$\Phi(x,t) = \int_0^t \varphi(x,s)ds, \ \forall x \in \overline{\Omega}, \ t \geq 0.$$

Problem (1) is a generalization of a model introduced by Kirchhoff [16], who studied the following equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0. \tag{3}$$

Problem (3) extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. Latter, the study of Kirchhoff type equations has already been extended to the case involving the p-Laplacian

$$-M\left(\int_\Omega |\nabla u|^p \, dx\right)\Delta_p u = f(x,u) \quad \text{in } \Omega$$

see [7, 13]. On the other hand, there is a great number of papers which have dealt with nonlocal p(x)-Laplacian equations, we refer the reader to [3, 8, 18] and the references therein for an overview on this subject.

We point out the fact that if $M(t) \equiv 1$, problem (1) becomes a nonlinear and non-homogeneous problem, which has been received considerable attention in recent years and studied by some authors in Orlicz-Sobolev spaces, see [1, 4, 5, 23] for the advances and references of this area. However, to our knowledge, there is not a great number of papers which have dealt with nonlocal and non-homogeneous equations through Orlicz-Sobolev spaces, we quoted some interesting papers [6, 12, 14]. In [12], Figueiredo et al. studied the existence of solutions for a class of nonlocal and non-homogeneous equations using Krasnoselskiis genus. In [14], the authors considered problem (1) in the special case when $f(x,u)=\lambda|u|^{q(x)-2}u$. In [6], the author studied the existence of solutions for the problem using a variational principle due to Ricceri [21]. Motivated by the contributions cited above, in this paper we study the existence of nontrivial solution for the nonlocal problem (1) with perturbation g in Orlicz-Sobolev spaces. Our proofs are essentially based on the mountain pass theorem combined with the Ekeland variational principle.

2. THE FUNCTIONAL RAMEWORK

Here, we state some interesting properties of the theory of Orlicz-Sobolev spaces that will be useful to discuss problem (1). To be more precise, for the function $\varphi(x,t)$ which satisfies condition $\mathrm{H}(\varphi)$, we assume that the function

$$\Phi(x,t)=\int_0^t\varphi(x,s)ds,\ x\in\overline{\Omega},\ t\geq 0$$

belongs to class Φ (see [20], p. 33), i.e., the function Φ satisfies the following conditions:

(Φ_1) for all $x\in\Omega$, $\varphi(x,.)\colon[0,+\infty]\to\mathbb{R}$ is a nondecreasing continuous function, with $\Phi(x,0)=0$ and $\Phi(x,t)>0$ whenever $t>0$, $\lim_{t\to\infty}\Phi(x,t)=+\infty$,

(Φ_2) for every $t>0$, $\Phi(.,t)\colon\Omega\to\mathbb{R}$ is a measurable function.

Since $\varphi(x,.)$ satisfies condition $\mathrm{H}(\varphi)$, we deduce that $\Phi(x,.)$ is convex and increasing from $\mathbb{R}^+\to\mathbb{R}^+$.

Now, for the function Φ introduced above, we define the generalized Orlicz space

$$L^{\Phi}(\Omega)=\left\{u:\Omega\to\mathbb{R}\text{, measurable; }\lim_{\lambda\to 0^+}\int_\Omega\Phi\left(x,\lambda|u(x)|\right)dx=0\right\}.$$

The space $L^{\Phi}(\Omega)$ is a Banach space endowed with the Luxemburg norm

$$\|u\|_{\Phi}=\inf\left\{\mu>0;\int_\Omega\Phi\left(x,\frac{|u(x)|}{\mu}\right)dx\leq 1\right\}$$

or the equivalent norm (the Orlicz norm)

$$|u|_{\Phi}=\sup\left\{\left|\int_\Omega uv\mathrm{dx}\right|,v\in L^{\overline{\Phi}}(\Omega),\int_\Omega\overline{\Phi}\left(x,|v(x)|\right)d\mathrm{x}\leq 1\right\},$$

where $\overline{\Phi}$ denotes the conjugate Young function of Φ, that is, for each $x\in\overline{\Omega}$ and $t\geq 0$,

$$\overline{\Phi}(x,t)=\sup_{s>0}\{ts-\Phi(x,s);s\in\mathbb{R}\}.$$

Furthermore, for Φ and $\overline{\Phi}$ conjugate Young functions, Holder’s inequality holds true

$$\left|\int_\Omega uvdx\right|\leq C.\|u\|_{\Phi}.\|v\|_{\overline{\Phi}},\ \forall u\in L^{\Phi}(\Omega),\ \forall v\in L^{\overline{\Phi}}(\Omega),$$

where C is a positive constant.

In this paper, we assume that there exist two positive constants α and β such that

$$1<\alpha\leq\frac{t\varphi(x,t)}{\Phi(x,t)}\leq\beta<+\infty,\quad\forall x\in\overline{\Omega},\quad t\geq 0.\quad(4)$$

The above relation implies that Φ satisfies the Δ_2-condition, i.e

$$\Phi(x,2t)\leq K.\Phi(x,t),\quad\forall x\in\overline{\Omega},\quad t\geq 0,\qquad(5)$$

where K is a positive constant.

Furthermore, we assume that Φ satisfies the following condition:

For each $x\in\overline{\Omega}$ the function $t\to\Phi\left(x,\sqrt{t}\right)$ is convex on $[0,+\infty)$. (6)

Relations (5) and (6) assure that $L^{\Phi}(\Omega)$ is an uniformly convex space and thus, a reflexive space.

Here, we give some relations between the norm $\|\cdot\|_\Phi$ and the modular:

$$\rho_\Phi(u) := \int_\Omega \Phi\left(x, |u(x)|\right) dx .$$

Proposition 2.1 ([19]). *Assume that (4), then the following relations hold:*

$$\|u\|_\Phi^\alpha \le \rho_\Phi(u) \le \|u\|_\Phi^\beta \qquad (7)$$

for all $u \in L^\Phi(\Omega)$ *with* $\|u\|_\Phi > 1$,

$$\|u\|_\Phi^\beta \le \rho_\Phi(u) \le \|u\|_\Phi^\alpha \qquad (8)$$

for all $u \in L^\Phi(\Omega)$ *with* $\|u\|_\Phi < 1$.

We denote by $W^{1,\Phi}(\Omega)$ the corresponding generalized Orlicz-Sobolev space for problem (1), defined by

$$W^{1,\Phi}(\Omega) = \left\{ u \in L^\Phi(\Omega);\ \frac{\partial u}{\partial x_i} \in L^\Phi(\Omega),\ i = 1, \dots, N \right\},$$

equipped with the equivalent norms

$$\|u\|_{1,\Phi} = \|\nabla u\|_\Phi + \|u\|_\Phi ,\ \|u\|_{2,\Phi} = \max\left\{ \|\nabla u\|_\Phi , \|u\|_\Phi \right\},$$

$$\|u\| = \inf\Big\{\mu > 0; \int_\Omega \Big[\Phi\Big(x, \frac{|u(x)|}{\mu}\Big) + \Phi\Big(x, \frac{|\nabla u(x)|}{\mu}\Big)\Big] dx \le 1\Big\}. \qquad (9)$$

More precisely, (see, e.g [19]) for every $u \in W^{1,\Phi}(\Omega)$ we have:

$$\|u\| \le 2\|u\|_{2,\Phi} \le 2\|u\|_{1,\Phi} \le 4\|u\|.$$

The generalized Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ endowed with one of the above norms is a reflexive Banach space. In the following, we will use the norm $\|.\|$ on $X := W^{1,\Phi}(\Omega)$.

Proposition 2.2 ([19]). *The following relations hold:*

$$\int_\Omega \left[\Phi\left(x, |u(x)|\right) + \Phi\left(x, |\nabla u(x)|\right)\right] dx \ge \|u\|^\alpha$$

for all $u \in X$ *with* $\|u\| > 1$,

$$\int_\Omega \left[\Phi\left(x, |u(x)|\right) + \Phi\left(x, |\nabla u(x)|\right)\right] dx \ge \|u\|^\beta$$

for all $u \in X$ *with* $\|u\| < 1$.

Remark 2.3. Assuming that Φ and Ψ belong to class Φ and there exists two positives constants k_1; k_2 and $\eta(x) \in L^1(\Omega)$, $\eta(x) \ge 0$ a.e. $u \in \Omega$ such that for all $x \in \overline{\Omega}, t \ge 0$,

$$\Psi(x,t) \le k_1 \Phi(x, k_2.t) + \eta(x) \ge 0, \qquad (10)$$

then there exists a continuous embedding $L^\Phi(\Omega) \subset L^\Psi(\Omega)$ (see [20, Theorem 8.5]). We point out that if (10) holds with $\inf_{x\in\Omega} \Phi(x,1) > 0$, $\inf_{x\in\Omega} \Psi(x,1) > 0$, then $W^{1,\Phi}(\Omega)$ is continuously embedded in $W^{1,\Psi}(\Omega)$.

In this paper, we study the problem (1) in the particular case when Φ satisfies:

$$M.|t|^{p(x)} \le \Phi(x,t),\ \forall x \in \overline{\Omega},\ t \ge 0, \qquad (11)$$

where $M > 0$ is a positive constant and the function $p(x) \in C_+(\overline{\Omega})$ with $1 < p^- := \inf_{x\in\Omega} p(x) \le p^+ := \max_{x\in\Omega} p(x) < N.$

Here, $C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h(x) > 1, \forall x \in \overline{\Omega}\}$.

We define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{, measuable: } \int_\Omega |u(x)|^{p(x)} dx < +\infty \right\}.$$

This space endowed with the Luxemburg norm,

$$\|u\|_{p(x)} = \inf\left\{ \tau > 0 : \int_\Omega \left| \frac{u(x)}{\tau} \right|^{p(x)} dx < 1 \right\}$$

is a separable and reflexive Banach space. Denoting by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$; for any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ we have the following Holder type inequality

$$\int_\Omega |uv|\, dx \le \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(x)} \|v\|_{p'(x)} . \qquad (12)$$

Now, we introduce the modular of the Lebesgue-Sobolev space $L^{p(x)}(\Omega)$ as mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$, defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx,\ \forall u \in L^{p(x)}(\Omega).$$

In the following proposition, we give some relations between the Luxemburg norm and the modular.

Proposition 2.4 ([10]). *If* $u, u_n \in L^{p(x)}(\Omega)$, *then following properties hold true:*

(1) $\|u\|_{p(x)} \leq 1 \Rightarrow \|u\|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^-}$;

(2) $\|u\|_{p(x)} \leq 1 \Rightarrow \|u\|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^+}$;

(3) $\lim_{n\to\infty}\|u_n\|_{p(x)} = 0 \Leftrightarrow \lim_{n\to\infty}\rho_{p(x)}(u_n) = 0$;

(4) $\lim_{n\to\infty}\|u_n\|_{p(x)} = +\infty \Leftrightarrow \lim_{n\to\infty}\rho_{p(x)}(u_n) = +\infty$.

Next, we define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : \ |\nabla u| \in L^{p(x)}(\Omega)\right\}.$$

endowed with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{p(x)} + \||\nabla u|\|_{p(x)},$$

The space $W^{1,p(x)}(\Omega)$ is separable and reflexive.

Proposition 2.5 ([10]). *For* $p, r \in C_+(\overline{\Omega})$ *such that* $r(x) \leq p^*(x)$ $(r(x) < p^*(x))$ *for all* $x \in \overline{\Omega}$, *there is a continuous (compact) embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega),$$

where

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & if \ \ p(x) < N \\ +\infty & if \ \ p(x) \geq N. \end{cases}$$

Before stating our results, we make the following assumptions on the functions $f(x,t)$ and $M(t)$ as follows:

(m_1) $M(t)$ is a nondecreasing continuous function, and there exists $m_0 > 0$ such that $M(t) \geq m_0$ for all $t \geq 0$.

(m_2) There exists $\sigma \in (\beta/q^-, 1)$ such that

$$\hat{M}(t) = \sigma M(t)t, \quad t \geq 0,$$

where $\hat{M}(t) = \int_0^t M(s)ds, \quad t \geq 0.$

(f_1) $\quad f(x,t) = o\left(|t|^{p(x)-1}\right)$ *as* $t \to 0,$ uniformly for $x \in \Omega$.

(f_2) $\quad f(x,t) = o\left(|t|^{q(x)-1}\right)$ *as* $t \to +\infty,$ uniformly for $x \in \Omega$, where $q(x) \in C_+(\overline{\Omega})$ such that $q(x) < p^*(x)$.

(f_3) there exists $\theta > \max\{p^-, q^+\}$ such that

$$\theta F(x,t) := \theta \int_0^t f(x,s)ds \leq t f(x,t)$$

$\forall t \in \mathbb{R}$ and $x \in \Omega$, where β, σ are given in (4) and assumption (m_2) respectively.

(f_4) $\quad \inf_{\{x\in\Omega:\ |t|=1\}} F(x,t) > 0.$

We denote by J the energy functional associated with problem (1), that is,

$$J(.) := I(.) - H(.),$$

where $I, H\colon\ X \to \mathbb{R}$ are defined as follows

$$I(u) = \widehat{M}(L(u)),$$

$$H(u) = \int_\Omega F(x,u)dx + \int_\Omega g(x)u dx, \qquad (13)$$

where L defined by (2). Then, $J \in C^1(X, \mathbb{R})$ and $u \in X$ is a weak solution of (1) if and only if u is a critical point of J. Moreover, we have

$$\langle J'(u), v\rangle = M(L(u)) \int_\Omega (a(x, |\nabla u|)\nabla u \nabla v + a(x,|u|)uv)dx - \int_\Omega f(x,u)v dx - \int_\Omega g(x)v dx,$$

for all $v \in X$.

We need the following lemma for the proofs of our main results.

Lemma 2.6. *If the condition (m_1) holds, then we have the following assertions:*

(i) *I is sequentially weakly lower semicontinuous and coercive;*

(ii) *$I': X \to X^*$ is strictly monotone;*

(iii) *I' is of type (S_+), i.e. if $u_n \rightharpoonup u$ weakly in X, and*

$$\overline{\lim_{n\to+\infty}} \left\langle I'(u_n) - I'(u), u_n - u\right\rangle = 0,$$

then $u_n \to u$ strongly in X.

Proof. (i) Since $\hat{M}(t) = M(t) \geq m_0 > 0$, $\hat{M}$ is an increasing function on $\mathbb{R}^+$. By using the fact that the the functional L defined by (2) is sequentially weakly lower semicontinuous (see [19]), we see that I is sequentially weakly lower semicontinuous. Obviously, thanks to Proposition 2.2 and *(m₁)*, for each $u \in X$ such that $\|u\| \geq 1$ we have

$$I(u) \geq m_0 L(u) \geq m_0 \|u\|^{\alpha}. \qquad (14)$$

So, I is coercive.

(ii) Consider the functional L, whose Gâteaux derivative at point $u \in X$ is given by

$$\left\langle L^{'}(u), v \right\rangle = \int_{\Omega} \left(a(x,|\nabla u|)\nabla u \nabla v + a(x,|u|uv \right) dx,$$

for all $v \in X$.

Taking into account [15, Lemma 3.2], $L^{'}$ is strictly monotone. So, by [24, Proposition 25.10], L is strictly convex. Moreover, since M is nondecreasing, $\widehat{M}$ is convex in $[0,+\infty]$. Thus, for every $u, v \in X$ with $u \neq v$, and every $s,t \in (0,1)$ with $s+t=1$, one has

$$\hat{M}\left(L(su+tv)\right) < \hat{M}\left(sL(u)+tL(v)\right) \leq s\hat{M}(L(u)) + t\hat{M}(L(v)).$$

From this, I is strictly convex, and, as already said, that $I^{'}$ is strictly monotone.

(iii) From (ii), if $u_n \rightharpoonup u$ as $n \to \propto$ in X and $\overline{\lim_{n\to+\infty}} \langle I'(u_n) - I'(u), u_n - u \rangle = 0$, we obtain

$$\lim_{n\to+\infty} \langle I'(u_n) - I'(u), u_n - u \rangle = 0,$$

we also have

$$\lim_{n\to+\infty} \langle I'(u_n), u_n - u \rangle = 0, \qquad (15)$$

which yields

$$\lim_{n\to+\infty} M(L(u_n)) \int_{\Omega} (a(x,|\nabla u_n|)\nabla u_n \nabla(u_n - u) + a(x,|u_n|)u_n(u_n - u))dx = 0.$$

Since $u_n \rightharpoonup u$ in X, it follows that $\{\|u_n\|\}$ is bounded sequence of real number. From the equivalent norms in relation (9), we see that $\{\|u_n\|_{\Phi}\}$ and $\{\||\nabla u_n|\|_{\Phi}\}$ are bounded sequences of real numbers. Then, Proposition 2.1 yields that the sequence $\{L(u_n)\}$ is bounded, up to subsequence, there is $t_0 \geq 0$ such that $L(u_n) \to t_0$. The fact that M is continuous,

$$M(L(u_n)) \to M(t_0) \geq m_0, \text{ as } n \to +\infty.$$

This and (16) imply

$$\lim_{n\to+\infty} \int_{\Omega} (a(x,|\nabla u_n|)\nabla u_n \nabla(u_n - u) + a(x,|u_n|)u_n(u_n - u))\, dx = 0. \qquad (17)$$

In the same way,

$$\lim_{n\to+\infty} \int_{\Omega} (a(x,|\nabla u|)\nabla u \nabla(u_n - u) + a(x,|u|)u(u_n - u))dx = 0. \quad (18)$$

Then, we obtain by using relations (17) and (18) that

$$o_n(1) = \int_{\Omega} (a(x,|\nabla u_n|)\nabla u_n - a(x,|\nabla u|)\nabla u)\nabla(u_n - u)dx + \int_{\Omega} (a(x,|u_n|)u_n - a(x,|u|)u)(u_n - u)dx. \qquad (19)$$

Using [17, Theorem 4] we obtain the strong convergence of $\{u_n\}$ in X, which ends the proof of (iii).

3. MAIN RESULTS AND PROOFS

Throughout the sequel and for simplicity, we use c_i (i = 1,2, ..), to denote the general nonnegative or positive constants. The first result of this paper can be described as follows.

Theorem 3.1. *Assume that (m₁), (f₁), (f₂) hold and suppose that* $q^+ < \alpha$. *Then, problem (1.1) has a weak solution, provided that* $g \in L^{p'(x)}(\Omega)$ *and* $g \not\equiv 0$.

Proof. By conditions *(f₁)* and *(f₂)*, it follows that for any $\varepsilon > 0$ there exists $c_\varepsilon = c(\varepsilon) > 0$ depending on ε such that

$$|F(x,t)| \leq \frac{\varepsilon}{p(x)}|t|^{p(x)} + \frac{C_\varepsilon}{q(x)}|t|^{q(x)} \qquad (20)$$

for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$.

Together with (m_1), and using Holder's inequality (12), we have

$$J(u) \geq \hat{M}\left(\int_\Omega \big(\Phi(x,|\nabla u|)+\Phi(x,|u|)\big)dx\right)-\frac{\varepsilon}{p^-}\int_\Omega |u|^{p(x)}\,dx$$
$$-\frac{c_\varepsilon}{q^-}\int_\Omega |u|^{q(x)}\,dx-\int_\Omega g(x)u(x)dx$$
$$\geq m_0\int_\Omega \big(\Phi(x,|\nabla u|)+\Phi(x,|u|)\big)dx-\frac{\varepsilon}{p^-}\int_\Omega |u|^{p(x)}\,dx$$
$$-\frac{c_\varepsilon}{q^-}\int_\Omega |u|^{q(x)}\,dx-c_1\|g\|_{p'(x)}\|u\|_{p(x)}.$$

Hence, for ε sufficiently small, it follows that

$$J(u) \geq \frac{m_0}{2}\int_\Omega(\Phi(x,|\nabla u|)+\Phi(x,|u|))dx - \frac{c_\varepsilon}{q^-}\int_\Omega |u|^{q(x)}dx - c_1\|g\|_{p'(x)}\|u\|_{p(x)}. \qquad (21)$$

By relation (11) and Remark 2.3 with $\Psi(x,t)=|t|^{p(x)}$, we deduce that the space X is continuously embedded in $W^{1,p(x)}(\Omega)$. On the other hand, Proposition 2.5 ensures that $W^{1,p(x)}(\Omega)$ is compactly embedded in $L^{q(x)}(\Omega)$. Thus, $X \hookrightarrow L^{q(x)}(\Omega)$ is compact. Then, there exist a positive constant c_2 such that

$$\|u\|_{q(x)} \leq c_2\|u\|, \quad \text{for all } u \in X. \qquad (22)$$

Then, by Propositions 2.2 and 2.4 the following hold

$$J(u) \geq m_0\, max\{\|u\|^\alpha, \|u\|^\beta\} - c_\varepsilon\, max\{\|u\|^{q^-}, \|u\|^{q^+}\} - c_4\|g\|_{p'(x)}\|u\| \to +\infty, \qquad (23)$$

as $\|u\| \to +\infty$ since $q^+ < \alpha$. By Lemma 2.6 (i), it is easy to verify that J is weakly lower semicontiguous. So J has a minimum point (see [22, Theorem 1.2]), which is a weak solution of problem (1).

Using the mountain pass theorem and Ekeland's variational principle, we obtain the second main result.

Theorem 3.2. *Assume that (m₁), (m₂), (f₁) - (f₄) hold and suppose that $\beta < q^-$ and $g \in L^{p'(x)}(\Omega)$, $g \not\equiv 0$. Then there exists a constant $\gamma > 0$ such that problem (1) admits at least two nontrivial different solutions $\underline{u}, \overline{u} \in X$ satisfying $J(\underline{u}) < 0 < J(\overline{u})$ provided that $\|g\|_{p'(x)} < \gamma$.*

We first prove the following auxiliary lemmas which will be used in the proof of Theorem 3.2.

Lemma 3.3. *Under the conditions (m_1), (f_1) and (f_2), there exist $a, \gamma,\ \varrho > 0$ such that $J(u) \geq a$ for any $u \in X$, $\|u\| = \varrho$ and for all $g \in L^{p'(x)}(\Omega)$ with $\|g\|_{p'(x)} \leq \gamma$.*

Proof. From relation (23), the following hold

$$J(u) \geq m_0 \max\left\{\|u\|^\alpha, \|u\|^\beta\right\} - c_3 \max\left\{\|u\|^{q^-}, \|u\|^{q^+}\right\} - c_4\|g\|_{p'(x)}\|u\|$$
$$= \begin{cases} \|u\|\left(m_0\|u\|^{\beta-1} - c_3\|u\|^{q^- -1} - c_4\|g\|_{L^{p'(x)}(\Omega)}\right), & u \in X \text{ with } \|u\| < 1 \\ \|u\|\left(m_0\|u\|^{\alpha-1} - c_3\|u\|^{q^+ -1} - c_4\|g\|_{L^{p'(x)}(\Omega)}\right), & u \in X \text{ with } \|u\| > 1. \end{cases}$$

Since $\beta < q^-$ (we also have $\alpha < q^+$), there exists $\varrho > 0$ such that

$$\max_{t\in \mathbb{R}^+} h(t) = h(\varrho) > 0,$$

where

$$h(t) := m_0 t^{\beta-1} - c_3 t^{q^- -1}.$$

Then, taking $\gamma = h(\varrho)/2c_4$ we obtain that $J(u) \geq a = \varrho h(\varrho)$ for $\|u\| = \varrho$ and for all $g \in L^{p'(x)}(\Omega)$ with $\|g\|_{p'(x)} \leq \gamma$.

Lemma 3.4. *Assume that conditions (m₂) and (f₃) hold. Then, there exists a nonnegative function $e \in X$ with $\|e\| > \varrho$ such that $J(e) < 0$, where is given in Lemma 3.3.*

Proof. Let

$$k(\tau) = \tau^{-\theta}F(x,\tau t) - F(x,t),\ t \geq 1.$$

We have

$$k'(t) = t^{-\theta-1}(f(x,\tau t)\tau t - F(x,\tau t)) \geq 0$$

for all $\tau \geq 1$ by (f_3). Hence, $k(\tau) \geq k(1)$ for all $\tau \geq 1$, that is,

$$F(x,\tau t) \geq \tau^\theta F(x,t) \qquad (24)$$

for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$ and $\tau \geq 1$.

Now, we show that

$$\Phi(x,\tau t) \leq \tau^\beta \Phi(x,t), \qquad (25)$$

for all $x \in \overline{\Omega}, t > 0$ and $\tau > 1$.

Indeed, from (4) for $\tau > 1$ be fixed, we have

$$ln(\Phi(x,\tau t)) - ln(\Phi(x,t)) = \int_0^{\tau t} \frac{\varphi(x,s)}{\Phi(x,s)} ds \le \int_t^{\tau t} \frac{\beta}{s} ds,$$

and it follows that relation (25) holds true. By the same way, integrating (m_2) we obtain

$$\widehat{M}(t) \le \frac{\widehat{M}(t_0)}{t_0^{\frac{1}{\sigma}}} = c_6 t^{1/\sigma}, \text{ for all } t \ge t_0 > 0. \quad (26)$$

Now, let a function $\psi \in C_0^\infty(\Omega), \psi \ge 0$, $\psi \not\equiv 0$ and $F(x,\psi) > 0$. For $u \in X\backslash\{0\}$ and $\tau > 1$, in view of relations (24), (25) and (26), we obtain

$$J(\tau\psi) = \widehat{M}\left(\int_\Omega (\Phi(x,\tau|\nabla\psi|) + \Phi(x,\tau|\psi|))\right) dx - \int_\Omega F(x,\tau\psi)dx - \int_\Omega g(x)\tau\psi dx$$
$$\le c_6\left(\int_\Omega (\Phi(x,\tau|\nabla\psi|) + \Phi(x,\tau|\psi|))dx\right)^{1/\sigma} - \tau^\theta \int_\Omega F(x,\psi)dx - \tau\int_\Omega g(x)\psi dx$$
$$\le c_6 \tau^{\beta/\sigma}\left(\int_\Omega (\Phi(x,|\nabla\psi|) + \Phi(x,|\psi|))\right)^{1/\sigma} - \tau^\theta \int_\Omega F(x,\psi)dx - \tau\int_\Omega g(x)\psi dx \quad (27)$$

since $1 < \beta < \beta/\sigma < q^- < \theta$, we deduce that $J(\tau\psi) \to -\infty$ as $\tau \to +\infty$. So Lemma 3.4 is proved by choosing $e = \tau_*\psi$ with $\tau_* > 0$ large enough such that $\|e\| > \varrho$.

Definition 3.5. We say that J satisfises the Palais-Smale condition at level $c \in \mathbb{R}$ (briefly (PS)c) on X, if any sequence $J\{u_n\} \subset X$, such that $J(u_n) \to c$ and $J'(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence.

Lemma 3.6. *Assume that conditions (m_1), (m_2) and $(f_1) - (f_3)$ hold. Then the functional J satisfies the $(PS)_c$ condition with $c \neq 0$.*

Proof. Consider a sequence $\{u_n\} \subset X$ which satisfies

$$J(u_n) \to \overline{c} > 0, J'(u_n) \to 0 \text{ as } n \to \infty. \quad (28)$$

Let us show that $\{u_n\}$ is bounded in X. Assume $\|u\| > 1$ for convenience, according to (m_1), (m_2), (f_3), (4) and Proposition 2.2, for n large enough, we have

$$1 + c_7 + \|u_n\| \ge J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n\rangle$$
$$\ge \sigma M\left(\int_\Omega (\Phi(x,|\nabla u_n|) + \Phi(x,|u_n|))\,dx\right)\int_\Omega (\Phi(x,|\nabla u_n|) + \Phi(x,|u_n|))\,dx$$
$$-\frac{1}{\theta}M\left(\int_\Omega (\Phi(x,|\nabla u_n|) + \Phi(x,|u_n|))\,dx\right)\int_\Omega (a(x,|\nabla u_n|)|\nabla u_n|^2 + a(|u_n|)u_n^2)dx$$
$$-\int_\Omega F(x,u_n)dx + \frac{1}{\theta}\int_\Omega f(x,u_n)u_n dx - \frac{\theta-1}{\theta}\int_\Omega g(x)u_n dx$$
$$\ge \sigma M\left(\int_\Omega (\Phi(x,|\nabla u_n|) + \Phi(x,|u_n|))dx\right)\int_\Omega (\Phi(x,|\nabla u_n|) + \Phi(x,|u_n|))dx$$
$$-\frac{1}{\theta}M\left(\int_\Omega (\Phi(x,|\nabla u_n|) + \Phi(x,|u_n|))dx\right)\int_\Omega (\varphi(x,|\nabla u_n|)|\nabla u_n| + \varphi(x,|u_n|)u_n)dx - \frac{c_4(\theta-1)}{\theta}\|g\|_{p'(x)}\|u_n\|$$
$$\ge \sigma M\left(\int_\Omega (\Phi(x,|\nabla u_n|) + \Phi(x,|u_n|))dx\right)\int_\Omega (\Phi(x,|\nabla u_n|) + \Phi(x,|u_n|))dx$$
$$-\frac{\beta}{\theta}M\left(\int_\Omega (\Phi(x,|\nabla u_n|) + \Phi(|u_n|))dx\right)\int_\Omega (\Phi(x,|\nabla u_n|) + \Phi(x,|u_n|))dx - c_8\|u_n\|$$
$$\ge m_0(\sigma - \beta/\theta)\|u_n\|^\alpha - c_8\|u_n\|.$$

Taking into account $\sigma > \beta/q^- > \beta/\theta$, we conclude that $\{u_n\}$ is bounded. For a subsequence we can assume that $u_n \rightharpoonup \bar{u}$ in X. Then $\langle J'(u_n), u_n - \overline{u}\rangle \to 0$, that is

$$M(L(u_n))\int_\Omega (a(x,|\nabla u_n|)\nabla u_n \nabla(u_n-\overline{u}) + a(x,|u_n|)u_n(u_n-\overline{u})dx - \int_\Omega f(x,u_n)(u_n-\overline{u})dx - \int_\Omega g(x)(u_n-\overline{u})dx \to 0.$$

From (f_1) and (f_2), using again Holder's inequality, it follows that

$$\int_\Omega f(x,u_n)(u_n-\overline{u})dx \to 0$$

and

$$\int_\Omega g(x)(u_n-\overline{u})dx \to 0.$$

Therefore, one has

$$M(L(u_n))\int_\Omega (a(x,|\nabla u_n|)\nabla u_n \nabla(u_n-\overline{u}) + a(x,|u_n|)u_n(u_n-\overline{u}))dx \to 0.$$

From Lemma 2.6 (iii), I' is of type (S_+), then $u_n \to \overline{u}$ strongly.

Proof of Theorem 3.2. The proof is divided into two steps:

Step 1: From Lemmas 3.3 and 3.4, by mountain pass theorem due to Ambrosetti and Rabinowitz [2], there exists a sequence $\{u_n\} \subset X$ such that

$$J(u_n) \to \overline{c} > 0, \quad J'(u_n) \to 0 \text{ as } n \to \infty. \quad (29)$$

By Lemma 3.6, that there exists $\overline{u} \in X$ such that $J(\overline{u}) = \overline{c} > 0, J'(\overline{u}) = 0$, i.e., $\overline{u}$ is a nontrivial weak solution of problem (1).

Step 2: For each $(x,t) \in \Omega \times \mathbb{R}$, set $h(\tau) = F(x,\tau^{-1}t)\tau^\theta, \tau \in [1,+\infty)$. By condition (f_3),

$$h'(\tau) = \tau^{\theta-1}[\theta F(x,\tau^{-1}t) - \tau^{-1}tf(z,\tau^{-1}t)] \le 0,$$

so $h(\tau)$ is nonincreasing. Thus, for any $|t| \ge 1$ we have $h(1) \ge h(|t|)$, that is,

$$F(x,t) \ge F(x,|t|^{-1}t)|t|^\theta \ge a_9|t|^\theta,$$

where $a_9 = inf_{\{x\in\Omega,|t|=1\}} F(x,t) > 0$ by (f_4).

From (f_1), there exists $\delta > 0$ such that

$$\frac{|f(x,t)t|}{|t|^{p(x)}} \le \frac{|f(x,t)|}{|t|^{p(x)-1}} \le 1$$

for all $x \in \Omega$ and $0 < |t| \le \delta$. By condition (f_2), for all $x \in \Omega$ and $\delta \le |t| \le 1$, there exists $c_{10} > 0$ such that

$$\frac{|f(x,t)t|}{|t|^{q(x)}} \le c_{10}\,.$$

Hence, for all $x \in \Omega$ and $0 \le |t| \le 1$, we have

$$f(x,t)t \ge -|t|^{p(x)} - c_{10}|t|^{q(x)}.$$

Using the equality $F(x,t) = \int_0^1 f(x,\tau t)d\tau$, it follows that

$$F(x,t) \ge -\frac{1}{p(x)}|t|^{p(x)} - \frac{c_{10}}{q(x)}|t|^{q(x)}$$

for all $x \in \Omega$ and all $0 \le |t| \le 1$. Therefore, we deduce that

$$F(x,t) \ge c_9|t|^\theta - \frac{1}{p(x)}|t|^{p(x)} - \frac{c_{10}}{q(x)}|t|^{q(x)}$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.

From the fact that $g \in L^{p'(x)}(\Omega)$ and $g \not\equiv 0$, we can choose a function $\varphi \in X$ such that

$$\int_\Omega g(x)\varphi(x)dx > 0.$$

Then, arising as (27) we obtain

$$J(\tau\varphi) = \widehat{M}\left(\int_\Omega (\Phi(x,\tau|\nabla\psi|) + \Phi(x,\tau|\psi|)\right)dx - \int_\Omega F(x,\tau\psi)dx - \int_\Omega g(x)\tau\psi dx$$
$$\le c_6\tau^{\beta/\sigma}\left(\int_\Omega (\Phi(x,|\nabla\varphi|) + \Phi(x,|\varphi|))dx\right)^{1/\sigma} - c_9\tau^\theta\int_\Omega |\varphi|^\theta dx + c_{11}\tau^{p^-}\int_\Omega |\varphi|^{p(x)}dx + c_{12}\tau^{q^-}\int_\Omega |\varphi|^{p(x)}dx - \tau\int_\Omega g(x)\varphi dx < 0,$$

for $\tau > 0$ small enough since $q^- > \beta/\sigma$ and $\theta > \max\{p^-,q^-\} > 1$. Thus, we obtain

$$-\infty < \underline{c} := inf_{\overline{B}_\varrho(0)} J < 0,$$

where ϱ is given by Lemma 3.3 and $B_\varrho(0) \subset X$ denote the ball centered at the origin and of radius ϱ.

Now, let us choose $\varepsilon > 0$ such that

$$\varepsilon < \inf_{\partial B_\varrho(0)} J - \inf_{B_\varrho(0)} J. \quad (30)$$

Applying Ekeland's variational principle to the functional $J: \overline{B}_\varrho(0) \to \mathbb{R}$, if follows that there exists $u_\varepsilon \in \overline{B}_\varrho(0)$ such that

$$\begin{cases} I(u_\varepsilon) < inf_{\overline{B}_\varrho(0)} J(u) + \varepsilon \\ J(u_\varepsilon) < J(u) + \varepsilon\|u-u_\varepsilon\|, \forall u \in \overline{B}_\varrho(0)\backslash\{u_\varepsilon\} \end{cases} \quad (31)$$

By (30) and the fact that

$$J(u_\varepsilon) < inf_{\overline{B}_\varrho(0)} J(u) + \varepsilon$$
$$< \inf_{B_\varrho(0)} J(u) + \varepsilon < \inf_{B_\varrho(0)} J(u),$$

it follows that $u_\varepsilon \in B_\varrho(0)$. From these facts, we have that u_ϵ is a local minimum of the functional $K(u) = J(u) + \varepsilon\|u - u_n\|$ defined from $\overline{B}_\varrho(0)$ onto $\mathbb{R}$. Therefore, for $v \in B_1(0)$ and sufficiently small $t > 0$, we have

$$0 \leq \frac{K(u_\varepsilon + tv) - K(u_\varepsilon)}{t}$$
$$= \frac{J(u_\varepsilon + tv) - J(u_\varepsilon)}{t} + \varepsilon\|v\|.$$

Letting $t \to 0^+$ it following that

$$\langle J'(u_\varepsilon), v\rangle + \varepsilon\|v\| \geq 0,$$

we infer that

$$\|J'(u_\varepsilon)\| \leq \varepsilon. \qquad (32)$$

From relations (31) and (32), there exists a sequence $\{u_n\} \subset B_\varrho(0)$ such that

$$J(u_n) \to \underline{c}, \quad J'(u_n) \to 0. \qquad (33)$$

In view of Lemma 3.6, $\{u_n\}$ is a bounded sequence in X. Thus, there exists $\underline{u} \in X$ such that, up to a subsequence, $\{u_n\}$ converges strongly to $\underline{u}$ and $J(\underline{u}) = \underline{c} < 0$, $J'(\underline{u}) = 0$, i.e., $\underline{u}$ is also a nontrivial weak solution for problem (1) such that $\underline{u} \neq \overline{u}$. The proof of Theorem 3.2 is now complete.

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SỰ TỒN TẠI NGHIỆM CHO MỘT LỚP PHƯƠNG TRÌNH KHÔNG THUẦN NHẤT VÀ KHÔNG ĐỊA PHƯƠNG TRONG KHÔNG GIAN ORLICZ-SOBOLEV

Tóm tắt. *Trong bài báo này, chúng tôi nghiên cứu sự tồn tại đa nghiệm cho một lớp bài toán không thuần nhất và không địa phương trong không gian Orlicz-Sobolev. Các kết quả của chúng tôi ở đây được thiết lập bằng cách dùng định lí qua núi kết hợp với nguyên lí biến phân Ekeland.*

Từ khóa: *Toán tử không thuần nhất; Không gian Orlicz-Sobolev; Bài toán kiểu Kirchhoff type; Phương pháp biến phân.*

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