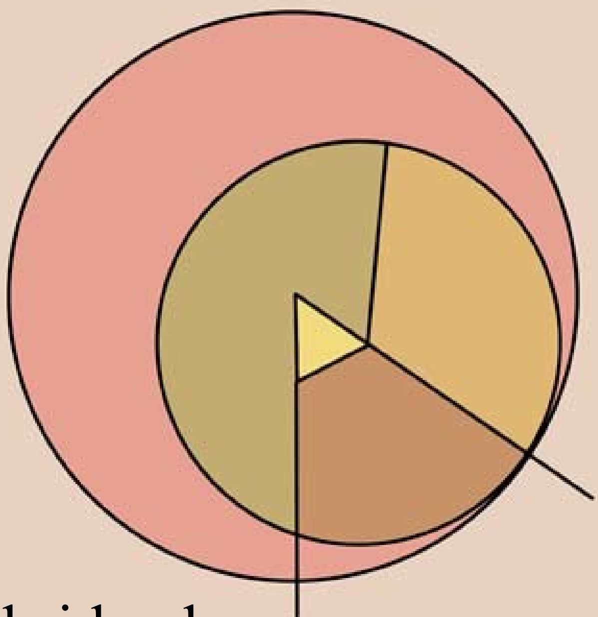
Vol.2(Books III-IX)

EUCLID

THE THIRTEEN BOOKS OF
THE ELEMENTS

Translated with introduction and commentary by Sir Thomas L. Heath



Second Edition Unabridged

THE THIRTEEN BOOKS

OF

EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

WITH INTRODUCTION AND COMMENTARY

BY

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VOLUME II

BOOKS III—IX

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THE THIRTEEN BOOKS

OF

EUCLID'S ELEMENTS

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CONTENTS OF VOLUME II

BOOK III.

DEFINITIONS.

1. **Equal circles** are those the diameters of which are equal, or the radii of which are equal.

2. A straight line is said to **touch a circle** which, meeting the circle and being produced, does not cut the circle.

3. **Circles** are said to **touch one another** which, meeting one another, do not cut one another.

4. In a circle straight lines are said **to be equally distant from the centre** when the perpendiculars drawn to them from the centre are equal.

5. And that straight line is said to be **at a greater distance** on which the greater perpendicular falls.

6. A **segment of a circle** is the figure contained by a straight line and a circumference of a circle.

7. An **angle of a segment** is that contained by a straight line and a circumference of a circle.

8. An **angle in a segment** is the angle which, when a point is taken on the circumference of the segment and straight lines are joined from it to the extremities of the straight line which is the **base of the segment,** is contained by the straight lines so joined.

9. And, when the straight lines containing the angle cut off a circumference, the angle is said to **stand upon** that circumference.

10. A **sector of a circle** is the figure which, when an angle is constructed at the centre of the circle, is contained by the straight lines containing the angle and the circumference cut off by them.

11. **Similar segments of circles** are those which admit equal angles, or in which the angles are equal to one another.

DEFINITION 1.

Ἴσοι κύκλοι εἰσίν, ὧν αἱ διάμετροι ἴσαι εἰσίν, ἢ ὧν αἱ ἐκ τῶν κέντρων ἴσαι εἰσίν.

Many editors have held that this should not have been included among definitions. Some, e.g. Tartaglia, would call it a *postulate*; others, e.g. Borelli and Playfair, would call it an *axiom*; others again, as Billingsley and Clavius, while admitting it as a *definition*, add explanations based on the mode of constructing a circle; Simson and Pfleiderer hold that it is a *theorem*. I think however that Euclid would have maintained that it is a definition in the proper sense of the term; and certainly it satisfies Aristotle's requirement that a "definitional statement" (ὁριστικὸς λόγος) should not only state the *fact* (τὸ ὅτι) but should indicate the *cause* as well (*De anima* II. 2, 413 a 13). The equality of circles with equal radii can of course be proved by superposition, but, as we have seen, Euclid avoided this method wherever he could, and there is nothing technically wrong in saying "By *equal circles* I mean circles with equal radii." No flaw is thereby introduced into the system of the *Elements*; for the definition could only be objected to if it could be proved that the equality predicated of the two circles in the definition was not the same thing as the equality predicated of other equal figures in the *Elements* on the basis of the Congruence-Axiom, and, needless to say, this cannot be proved because it is not true. The *existence* of equal circles (in the sense of the definition) follows from the existence of equal straight lines and I. Post. 3.

The Greeks had no distinct word for *radius*, which is with them, as here, *the (straight line drawn) from the centre* ἡ ἐκ τοῦ κέντρου (εὐθεῖα); and so definitely was the expression appropriated to the radius that ἐκ τοῦ κέντρου was used without the article as a predicate, just as if it were one word. Thus, e.g., in III. 1 ἐκ κέντρου γάρ means "for they are radii": cf. Archimedes, *On the Sphere and Cylinder* II. 2, ἡ BE ἐκ τοῦ κέντρου ἐστὶ τοῦ...κύκλου, *BE is a radius of the circle*.

DEFINITION 2.

Εὐθεῖα κύκλου ἐφάπτεσθαι λέγεται, ἥτις ἁπτομένη τοῦ κύκλου καὶ ἐκβαλλομένη οὐ τέμνει τὸν κύκλον.

Euclid's phraseology here shows the regular distinction between ἅπτεσθαι and its compound ἐφάπτεσθαι, the former meaning "to *meet*" and the latter "to *touch*." The distinction was generally observed by Greek geometers from Euclid onwards. There are however exceptions so far as ἅπτεσθαι is concerned; thus it means "to *touch*" in Eucl. IV. Def. 5 and sometimes in Archimedes. On the other hand, ἐφάπτεσθαι is used by Aristotle in certain cases where the orthodox geometrical term would be ἅπτεσθαι. Thus in *Meteorologica* III. 5 (376 b 9) he says a certain circle *will pass through all the angles* (ἁπασῶν ἐφάψεται τῶν γωνιῶν), and (376 a 6) M *will lie on a given (circular) circumference* (δεδομένης περιφερείας ἐφάψεται τὸ M). We shall find ἅπτεσθαι used in these senses in Book IV. Deff. 2, 6 and Deff. 1, 3 respectively. The latter of the two expressions quoted from Aristotle means that *the locus of* M *is a given circle*, just as in Pappus ἅψεται τὸ σημεῖον θέσει δεδομένης εὐθείας means that *the locus of* the point is a straight line given in position.

DEFINITION 3.

Κύκλοι ἐφάπτεσθαι ἀλλήλων λέγονται οἵτινες ἁπτόμενοι ἀλλήλων οὐ τέμνουσιν ἀλλήλους.

Todhunter remarks that different opinions have been held as to what is, or should be, included in this definition, one opinion being that it only means that the circles do not cut in the neighbourhood of the point of contact, and that it must be shown that they do not cut elsewhere, while another opinion is that the definition means that the circles do not cut at all. Todhunter thinks the latter opinion correct. I do not think this is proved; and I prefer to read the definition as meaning simply that the circles meet at a point but do not cut *at that point*. I think this interpretation preferable for the reason that, although Euclid does practically assume in III. 11—13, without stating, the theorem that circles touching at one point do not intersect anywhere else, he has given us, before reaching that point in the Book, means for proving for ourselves the truth of that statement. In particular, he has given us the propositions III. 7, 8 which, taken as a whole, give us more information as to the general nature of a circle than any other propositions that have preceded, and which can be used, as will be seen in the sequel, to solve any doubts arising out of Euclid's unproved assumptions. Now, as a matter of fact, the propositions are not used in any of the genuine proofs of the theorems in Book III.; III. 8 is required for the second proof of III. 9 which Simson selected in preference to the first proof, but the first proof only is regarded by Heiberg as genuine. Hence it would not be easy to account for the appearance of III. 7, 8 at all unless as affording means of answering possible *objections* (cf. Proclus' explanation of Euclid's reason for inserting the second part of I. 5).

External and *internal* contact are not distinguished in Euclid until III. 11, 12, though the *figure* of III. 6 (not the *enunciation* in the original text) represents the case of internal contact only. But the definition of touching circles here given must be taken to imply so much about *internal* and *external* contact respectively as that (*a*) a circle touching another internally must, immediately before "meeting" it, have passed through points *within* the circle that it touches, and (*b*) a circle touching another externally must, immediately before meeting it, have passed through points *outside* the circle which it touches. These facts must indeed be admitted if *internal* and *external* are to have any meaning at all in this connexion, and they constitute a minimum admission necessary to the proof of III. 6.

DEFINITION 4.

Ἐν κύκλῳ ἴσον ἀπέχειν ἀπὸ τοῦ κέντρου εὐθεῖαι λέγονται, ὅταν αἱ ἀπὸ τοῦ κέντρου ἐπ' αὐτὰς κάθετοι ἀγόμεναι ἴσαι ὦσιν.

DEFINITION 5.

Μεῖζον δὲ ἀπέχειν λέγεται, ἐφ' ἣν ἡ μείζων κάθετος πίπτει.

DEFINITION 6.

Τμῆμα κύκλου ἐστὶ τὸ περιεχόμενον σχῆμα ὑπό τε εὐθείας καὶ κύκλου περιφερείας.

DEFINITION 7.

Τμήματος δὲ γωνία ἐστὶν ἡ περιεχομένη ὑπό τε εὐθείας καὶ κύκλου περιφερείας.

This definition is only interesting historically. The *angle of a segment*, being the "angle" formed by a straight line and a "circumference," is of the kind described by Proclus as "mixed." A particular "angle" of this sort is the "*angle of a semicircle*," which we meet with again in III. 16, along with the so-called "horn-like angle" (*κερατοειδής*), the supposed "angle" between a tangent to a circle and the circle itself. The "angle of a semicircle" occurs once in Pappus (VII. p. 670, 19), but it there means scarcely more than the *corner* of a semicircle regarded as a point to which a straight line is directed. Heron does not give the definition of the *angle of a segment*, and we may conclude that the mention of it and of the *angle of a semicircle* in Euclid is a survival from earlier text-books rather than an indication that Euclid considered either to be of importance in elementary geometry (cf. the note on III. 16 below).

We have however, in the note on I. 5 above (Vol. I. pp. 252—3), seen evidence that the *angle of a segment* had played some part in geometrical proofs up to Euclid's time. It would appear from the passage of Aristotle there quoted (*Anal. prior.* I. 24, 41 b 13 sqq.) that the theorem of I. 5 was, in the text-books immediately preceding Euclid, proved by means of the equality of the two "*angles of*" any one segment. This latter property must therefore have been regarded as more elementary (for whatever reason) than the theorem of I. 5; indeed the definition as given by Euclid practically implies the same thing, since it speaks of only *one* "angle of a segment," namely "*the* angle contained by a straight line and a circumference of a circle." Euclid abandoned the actual use of the "angle" in question, but no doubt thought it unnecessary to break with tradition so far as to strike the definition out also.

DEFINITION 8.

Ἐν τμήματι δὲ γωνία ἐστίν, ὅταν ἐπὶ τῆς περιφερείας τοῦ τμήματος ληφθῇ τι σημεῖον καὶ ἀπ' αὐτοῦ ἐπὶ τὰ πέρατα τῆς εὐθείας, ἥ ἐστι βάσις τοῦ τμήματος, ἐπιζευχθῶσιν εὐθεῖαι, ἡ περιεχομένη γωνία ὑπὸ τῶν ἐπιζευχθεισῶν εὐθειῶν.

DEFINITION 9.

Ὅταν δὲ αἱ περιέχουσαι τὴν γωνίαν εὐθεῖαι ἀπολαμβάνωσί τινα περιφέρειαν, ἐπ' ἐκείνης λέγεται βεβηκέναι ἡ γωνία.

DEFINITION 10.

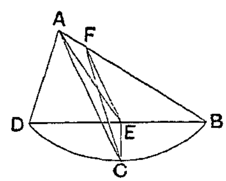
Τομεὺς δὲ κύκλου ἐστίν, ὅταν πρὸς τῷ κέντρῳ τοῦ κύκλου συσταθῇ γωνία, τὸ περιεχόμενον σχῆμα ὑπό τε τῶν τὴν γωνίαν περιεχουσῶν εὐθειῶν καὶ τῆς ἀπολαμβανομένης ὑπ' αὐτῶν περιφερείας.

A scholiast says that it was the *shoemaker's knife*, *σκυτοτομικὸς τομεύς*, which suggested the name *τομεύς* for a sector of a circle. The derivation of the name from a resemblance of shape is parallel to the use of *ἄρβηλος* (also a *shoemaker's knife*) to denote the well known figure of the Book of Lemmas partly attributed to Archimedes.

A wider definition of a sector than that given by Euclid is found in a Greek scholiast (Heiberg's Euclid, Vol. v. p. 260) and in an-Nairīzī (ed. Curtze, p. 112). "There are two varieties of sectors; the one kind have the angular vertices at the centres, the other at the circumferences. Those others which have their vertices neither at the circumferences nor at the centres, but at some other points, are for that reason not called sectors but sector-like figures (*τομοειδῆ σχήματα*)." The exact agreement between the scholiast and an-Nairīzī suggests that Heron was the authority for this explanation.

The *sector-like figure* bounded by an arc of a circle and two lines drawn from its extremities to meet at any point actually appears in Euclid's book *On divisions* (*περὶ διαιρέσεων*) discovered in an Arabic MS. and edited by Woepcke (cf. Vol. I. pp. 8—10 above). This treatise, alluded to by Proclus, had for its object the division of figures such as triangles, trapezia, quadrilaterals and circles, by means of straight lines, into parts equal or in given ratios. One proposition e.g. is, *To divide a triangle into two equal parts by a straight line passing through a given point on one side.* The proposition (28) in which the *quasi-sector* occurs is, *To divide such a figure by a straight line into two equal parts.* The solution in this case is given by Cantor (*Gesch. d. Math.* I_3, pp. 287—8).

If $ABCD$ be the given figure, E the middle point of BD and EC at right angles to BD, the broken line AEC clearly divides the figure into two equal parts.



Join AC, and draw EF parallel to it meeting AB in F.

Join CF, when it is seen that CF divides the figure into two equal parts.

DEFINITION 11.

Ὅμοια τμήματα κύκλων ἐστὶ τὰ δεχόμενα γωνίας ἴσας, ἢ ἐν οἷς αἱ γωνίαι ἴσαι ἀλλήλαις εἰσίν.

De Morgan remarks that the use of the word *similar* in "similar segments" is an anticipation, and that similarity *of form* is meant. He adds that the definition is a theorem, or would be if "similar" had taken its final meaning.

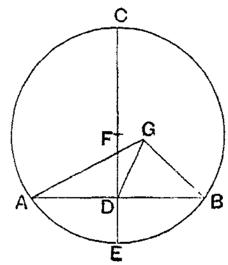
BOOK III. PROPOSITIONS.

PROPOSITION 1.

To find the centre of a given circle.

Let ABC be the given circle;

thus it is required to find the centre of the circle ABC.



Let a straight line AB be drawn through it at random, and let it be bisected at the point D;

from D let DC be drawn at right angles to AB and let it be drawn through to E;

let CE be bisected at F;

I say that F is the centre of the circle ABC.

For suppose it is not, but, if possible, let G be the centre,

and let GA, GD, GB be joined.

Then, since AD is equal to DB,

and DG is common,

the two sides AD, DG are equal to the two sides BD, DG respectively;

and the base GA is equal to the base GB, for they are radii;

therefore the angle ADG is equal to the angle GDB. [I. 8]

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; [I. Def. 10]

therefore the angle GDB is right.

But the angle FDB is also right;

therefore the angle FDB is equal to the angle GDB, the greater to the less: which is impossible.

Therefore G is not the centre of the circle ABC.

Similarly we can prove that neither is any other point except F.

Therefore the point F is the centre of the circle ABC.

PORISM. From this it is manifest that, if in a circle a straight line cut a straight line into two equal parts and at right angles, the centre of the circle is on the cutting straight line.

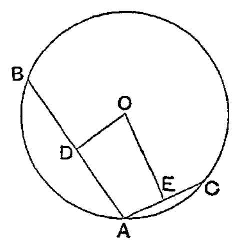
Q. E. F.

12. **For suppose it is not.** This is expressed in the Greek by the two words Μὴ γάρ, but such an elliptical phrase is impossible in English.

17. **the two sides AD, DG are equal to the two sides BD, DG respectively.** As before observed, Euclid is not always careful to put the equals in corresponding order. The text here has "*GD*, *DB*."

Todhunter observes that, when, in the construction, DC is said to be *produced* to E, it is assumed that D is within the circle, a fact which Euclid first demonstrates in III. 2. This is no doubt true, although the word διήχθω, "let it be *drawn through*," is used instead of ἐκβεβλήσθω, "let it be *produced*." And, although it is not necessary to assume that D is within the circle, it is necessary for the success of the construction that the straight line drawn through D at right angles to AB shall meet the circle in two points (and no more): an assumption which we are not entitled to make on the basis of what has gone before only.

Hence there is much to be said for the alternative procedure recommended by De Morgan as preferable to that of Euclid. De Morgan would first prove the fundamental theorem that "the line which bisects a chord perpendicularly must contain the centre," and then make III. 1, III. 25 and IV. 5 immediate corollaries of it. The fundamental theorem is a direct consequence of the theorem that, if P is *any* point equidistant from A and B, then P lies on the straight line bisecting AB perpendicularly. We then take any two chords AB, AC of the given circle and draw DO, EO bisecting them perpendicularly. Unless BA, AC are in one straight line, the straight lines DO, EO must meet in some point O (see note on IV. 5 for possible methods of proving this). And, since both DO, EO must contain the centre, O must be the centre.



This method, which seems now to be generally preferred to Euclid's, has the advantage of showing that, in order to find the centre of a circle, it is sufficient to know three points on the circumference. If therefore two circles have three points in common, they must have the same centre and radius, so that two circles cannot have three points in common without coinciding entirely. Also, as indicated by De Morgan, the same construction enables us (1) to draw the complete circle of which a segment or arc only is given (III. 25), and (2) to circumscribe a circle to any triangle (IV. 5).

But, if the Greeks had used this construction for finding the centre of a circle, they would have considered it necessary to add a proof that no other point than that obtained by the construction can be the centre, as is clear both from the similar *reductio ad absurdum* in III. 1 and also from the fact that Euclid thinks it necessary to prove as a separate theorem (III. 9) that, if a point within a circle be such that three straight lines (at least) drawn from it to the circumference are equal, that point must be the centre. In fact, however, the proof amounts to no more than the remark that the two perpendicular bisectors can have no more than one point common.

And even in De Morgan's method there is a yet unproved assumption. In order that DO, EO may meet, it is necessary that AB, AC should not be in one straight line or, in other words, that BC should not pass through A. This results from III. 2, which therefore, strictly speaking, should precede.

To return to Euclid's own proposition III. 1, it will be observed that the demonstration only shows that the centre of the circle cannot lie on either side of CD, so that it must lie on CD or CD produced. It is however taken for granted rather than proved that the centre must be the middle point of CE. The proof of this by *reductio ad absurdum* is however so obvious as to be scarcely worth giving. The same consideration which would prove it may be used to show that *a circle cannot have more than one centre*, a proposition which, if thought necessary, may be added to III. 1 as a corollary.

Simson observed that the proof of III. 1 could not but be by *reductio ad absurdum*. At the beginning of Book III. we have nothing more to base the proof upon than the definition of a circle, and this cannot be made use of unless we assume some point to be the centre. We cannot however assume that the point found by the construction is the centre, because that is the thing to be proved. Nothing is therefore left to us but to assume that some other point is the centre and then to prove that, whatever other point is taken, an absurdity results; whence we can infer that the point found is the centre.

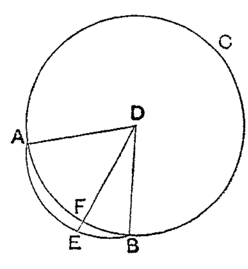
The Porism to III. 1 is inserted, as usual, parenthetically before the words ὅπερ ἔδει ποιῆσαι, which of course refer to the problem itself.

PROPOSITION 2.

If on the circumference of a circle two points be taken at random, the straight line joining the points will fall within the circle.

Let ABC be a circle, and let two points A, B be taken at random on its circumference;

I say that the straight line joined from A to B will fall within the circle.



For suppose it does not, but, if possible, let it fall outside, as AEB; let the centre of the circle ABC be taken [III. 1], and let it be D; let DA, DB be joined, and let DFE be drawn through.

Then, since DA is equal to DB,

the angle DAE is also equal to the angle DBE. [I. 5]

And, since one side AEB of the triangle DAE is produced,

the angle DEB is greater than the angle DAE. [I. 16]

But the angle DAE is equal to the angle DBE;

therefore the angle DEB is greater than the angle DBE.

And the greater angle is subtended by the greater side; [I. 19]

therefore DB is greater than DE.

But DB is equal to DF;

therefore DF is greater than DE,

the less than the greater: which is impossible.

Therefore the straight line joined from A to B will not fall outside the circle.

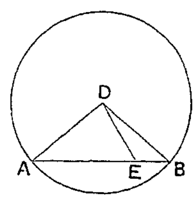
Similarly we can prove that neither will it fall on the circumference itself;

therefore it will fall within.

Therefore etc.

Q. E. D.

The *reductio ad absurdum* form of proof is not really necessary in this case, and it has the additional disadvantage that it requires the destruction of two hypotheses, namely that the chord is (1) outside, (2) on the circle. To prove the proposition directly, we have only to show that, if E be any point on the straight line AB between A and B, DE is less than the radius of the circle. This may be done by the method shown above, under I. 24, for proving what is assumed in that proposition, namely that, in the figure of the proposition, F falls below EG if DE is not greater than DF. The assumption amounts to the following proposition, which De Morgan would make to precede I. 24: "Every straight line drawn from the vertex of a triangle to the base is less than the greater of the two sides, or than either if they be equal." The case here is that in which the two sides are equal; and, since the angle DAB is equal to the angle DBA, while the exterior angle DEA is greater than the interior and opposite angle DBA, it follows that the angle DEA is greater than the angle DAE, whence DE must be less than DA or DB.



Camerer points out that we may add to this proposition the further statement that all points on AB *produced* in either direction are outside the circle. This follows from the proposition (also proved by means of the theorems that the exterior angle of a triangle is greater than either of the interior and opposite angles and that the greater angle is subtended by the greater side) which De Morgan proposes to introduce after I. 21, namely,

"The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line, and of others that which is nearer to the perpendicular is less than the more remote, and the converse; also not more than two equal straight lines can be drawn from the point to the line, one on each side of the perpendicular."

The fact that not more than two equal straight lines can be drawn from a given point to a given straight line not passing through it is proved by Proclus on I. 16 (see the note to that proposition) and can alternatively be proved by means of I. 7, as shown above in the note on I. 12. It follows that

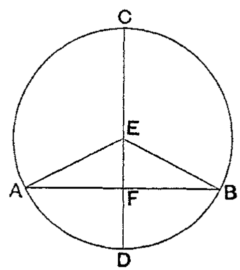
A straight line cannot cut a circle in more than two points:

a proposition which De Morgan would introduce here after III. 2. The proof given does not apply to a straight line *passing through the centre*; but that such a line only cuts the circle in two points is self-evident.

Proposition 3.

If in a circle a straight line through the centre bisect a straight line not through the centre, it also cuts it at right angles; and if it cut it at right angles, it also bisects it.

Let ABC be a circle, and in. it let a straight line CD through the centre bisect a straight line AB not through the centre at the point F;



I say that it also cuts it at right angles.

For let the centre of the circle ABC be taken, and let it be E; let EA, EB be joined.

Then, since AF is equal to FB,

and FE is common,

two sides are equal to two sides;

and the base EA is equal to the base EB;

therefore the angle AFE is equal to the angle BFE. [I. 8]

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; [I. Def. 10]

therefore each of the angles AFE, BFE is right.

Therefore CD, which is through the centre, and bisects AB which is not through the centre, also cuts it at right angles.

Again, let CD cut AB at right angles;

I say that it also bisects it, that is, that AF is equal to FB.

For, with the same construction,

since EA is equal to EB,

the angle EAF is also equal to the angle EBF. [I. 5]

But the right angle AFE is equal to the right angle BFE, therefore EAF, EBF are two triangles having two angles equal to two angles and one side equal to one side, namely EF, which is common to them, and subtends one of the equal angles;

therefore they will also have the remaining sides equal to the remaining sides; [I. 26]

therefore AF is equal to FB.

Therefore etc.

Q. E. D.

26. with the same construction, τῶν αὐτῶν κατασκευασθέντων.

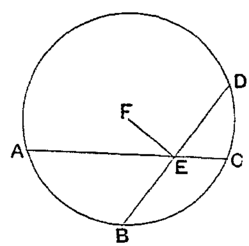
This proposition asserts the two *partial* converses (cf. note on I. 6) of the Porism to III. I. De Morgan would place it next to III. I.

Proposition 4.

If in a circle two straight lines cut one another which are not through the centre, they do not bisect one another.

Let $ABCD$ be a circle, and in it let the two straight lines AC, BD, which are not through the centre, cut one another at E;

I say that they do not bisect one another.



For, if possible, let them bisect one another, so that AE is equal to EC, and BE to ED;

let the centre of the circle $ABCD$ be taken [III. I], and let it be F; let FE be joined.

Then, since a straight line FE through the centre bisects a straight line AC not through the centre,

it also cuts it at right angles; [III. 3]

therefore the angle FEA is right.

Again, since a straight line FE bisects a straight line BD,

it also cuts it at right angles; [III. 3]

therefore the angle FEB is right.

But the angle FEA was also proved right;

therefore the angle FEA is equal to the angle FEB, the less to the greater: which is impossible.

Therefore AC, BD do not bisect one another.

Therefore etc.

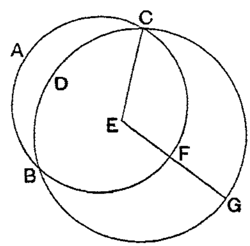
Q. E. D.

PROPOSITION 5.

If two circles cut one another, they will not have the same centre.

For let the circles ABC, CDG cut one another at the points B, C;

I say that they will not have the same centre.



For, if possible, let it be E; let EC be joined, and let EFG be drawn through at random.

Then, since the point E is the centre of the circle ABC,

EC is equal to EF. [I. Def. 15]

Again, since the point E is the centre of the circle CDG,

EC is equal to EG.

But EC was proved equal to EF also;

therefore EF is also equal to EG, the less to the greater: which is impossible.

Therefore the point E is not the centre of the circles ABC, CDG.

Therefore etc.

Q. E. D.

The propositions III. 5, 6 could be combined in one. It makes no difference whether the circles cut, or meet without cutting, so long as they do not coincide altogether; in either case they cannot have the same centre. The two cases are covered by the enunciation: *If the circumferences of two circles meet at a point they cannot have the same centre.* On the other hand, *If two circles have the same centre and one point in their circumferences common, they must coincide altogether.*