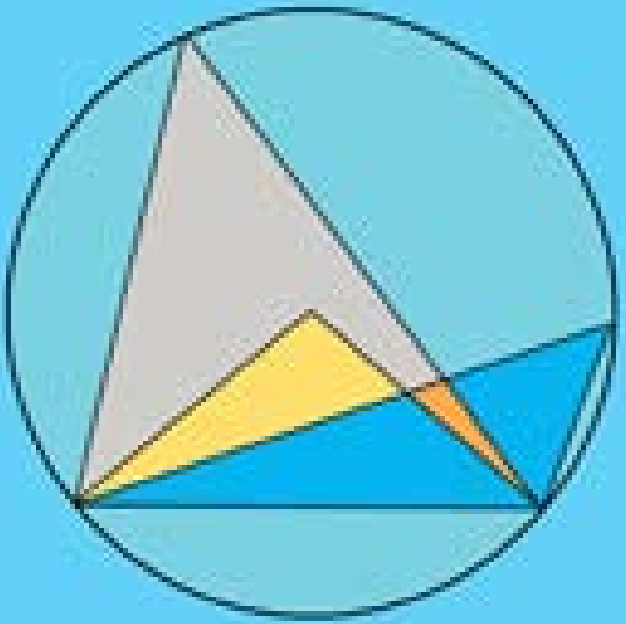
Vol.1(Books I and II)

EUCLID

THE THIRTEEN BOOKS OF THE ELEMENTS

Translated with introduction and commentary by Sir Thomas L. Heath



Second Edition Unabridged

THE THIRTEEN BOOKS
OF
EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

WITH INTRODUCTION AND COMMENTARY

BY

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VOLUME I

INTRODUCTION AND BOOKS I, II

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ERRATA

Vol. I. p. 19, line 17, *for* "but not a platform and sixpence" *read* "but not a figure and sixpence"

,, p. 105, line 10
,, p. 415, col. 2, line 17
Vol. II. p. 41, line 27
Vol. III. p. 539, col. 1, line 45
} *for* Christoph Schlüssel *read* Christoph Klau

Vol. I. p. 106, line 1
,, p. 415, col. 1, line 21
Vol. III. p. 538, col. 2, line 25
} *for* Giacomo *read* Giovanni

Vol. I. p. 182, line 21, *for* Ορος *read* Ὅρος

Vol. III. p. 499. In the figure, CV should be a *thick* line.

PREFACE

"THERE never has been, and till we see it we never shall believe that there can be, a system of geometry worthy of the name, which has any material departures (we do not speak of *corrections* or *extensions* or *developments*) from the plan laid down by Euclid." De Morgan wrote thus in October 1848 (*Short supplementary remarks on the first six Books of Euclid's Elements* in the *Companion to the Almanac* for 1849); and I do not think that, if he had been living to-day, he would have seen reason to revise the opinion so deliberately pronounced sixty years ago. It is true that in the interval much valuable work has been done on the continent in the investigation of the first principles, including the formulation and classification of axioms or postulates which are necessary to make good the deficiencies of Euclid's own explicit postulates and axioms and to justify the further assumptions which he tacitly makes in certain propositions, content apparently to let their truth be inferred from observation of the figures as drawn; but, once the first principles are disposed of, the body of doctrine contained in the recent text-books of elementary geometry does not, and from the nature of the case cannot, show any substantial differences from that set forth in the *Elements*. In England it would seem that far less of scientific value has been done; the efforts of a multitude of writers have rather been directed towards producing alternatives for Euclid which shall be more suitable, that is to say, easier, for schoolboys. It is of course not surprising that, in these days of short cuts, there should have arisen a movement to get rid of Euclid and to substitute a "royal road to geometry"; the marvel is that a book which was not written for schoolboys but for grown men (as all internal evidence shows, and in particular the essentially theoretical character of the work and its aloofness from anything of the nature of "practical" geometry) should have held its own as a school-book for so long. And now that Euclid's proofs and arrangement are no longer required from candidates at examinations there has been a rush of competitors anxious to be first in the field with a new text-book on the more "practical" lines which now find so much favour. The natural desire of each teacher who writes such a text-book is to give prominence to some special nostrum which he has found successful with pupils. One result is, too often, a loss of a due sense of proportion; and, in any case, it is inevitable that there should be great diversity of treatment. It was with reference to such a danger that Lardner wrote in 1846: "Euclid once superseded, every teacher would esteem his own work the best, and every school would have its own class book. All that rigour and exactitude which have so long excited the admiration of men of science would be at an end. These very words would lose all definite meaning. Every school would have a different standard; matter of assumption in one being matter of demonstration in another; until, at length, GEOMETRY, in the ancient sense of the word, would be altogether frittered away or be only considered as a particular application of Arithmetic and Algebra." It is, perhaps, too early yet to prophesy what will be the ultimate outcome of the new order of things; but it would at least seem possible that history will repeat itself and that, when chaos has come again in geometrical teaching, there will be a return to Euclid more or less complete for the purpose of standardising it once more.

But the case for a new edition of Euclid is independent of any controversies as to how geometry shall be taught to schoolboys. Euclid's work will live long after all the text-books of the present day are superseded and forgotten. It is one of the noblest monuments of antiquity; no mathematician worthy of the name can afford not to know Euclid, the real Euclid as distinct from any revised or rewritten versions which will serve for schoolboys or engineers. And, to know Euclid, it is necessary to know his language, and, so far as it can be traced, the history of the "elements" which he collected in his immortal work.

This brings me to the *raison d'être* of the present edition. A new translation from the Greek was necessary for two reasons. First, though some time has elapsed since the appearance of Heiberg's definitive text and prolegomena, published between 1883 and 1888, there has not been, so far as I know, any attempt to make a faithful translation from it into English even of the Books which are commonly read. And, secondly, the other Books, VII. to X. and XIII., were not included by Simson and the editors who followed him, or apparently in any English translation since Williamson's (1781—8), so that they are now practically inaccessible to English readers in any form.

In the matter of notes, the edition of the first six Books in Greek and Latin with notes by Camerer and Hauber (Berlin, 1824—5) is a perfect mine of information. It would have been practically impossible to make the notes more exhaustive at the time when they were written. But the researches of the last thirty or forty years into the history of mathematics (I need only mention such names as those of Bretschneider, Hankel, Moritz Cantor, Hultsch, Paul Tannery, Zeuthen, Loria, and Heiberg) have put the whole subject upon a different plane. I have endeavoured in this edition to take account of all the main results of these researches up to the present date. Thus, so far as the geometrical Books are concerned, my notes are intended to form a sort of dictionary of the history of elementary geometry, arranged according to subjects; while the notes on the arithmetical Books VII.—IX. and on Book X. follow the same plan.

I desire to express here my thanks to my brother, Dr R. S. Heath, Vice-Principal of Birmingham University, for suggestions on the proof sheets and, in particular, for the reference to the parallelism between Euclid's definition of proportion and Dedekind's theory of irrationals, to Mr R. D. Hicks for advice on a number of difficult points of translation, to Professor A. A. Bevan for help in the transliteration of Arabic names, and to the Curators and Librarian of the Bodleian Library for permission to reproduce, as frontispiece, a page from the famous Bodleian MS. of the *Elements*. Lastly, my best acknowledgments are due to the Syndics of the Cambridge University Press for their ready acceptance of the work, and for the zealous and efficient coöperation of their staff which has much lightened the labour of seeing the book through the Press.

T. L. H.

November, 1908.

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INTRODUCTION.

CHAPTER I.

EUCLID AND THE TRADITIONS ABOUT HIM.

As in the case of the other great mathematicians of Greece, so in Euclid's case, we have only the most meagre particulars of the life and personality of the man.

Most of what we have is contained in the passage of Proclus' summary relating to him, which is as follows[1]:

"Not much younger than these (sc. Hermotimus of Colophon and Philippus of Mende) is Euclid, who put together the Elements, collecting many of Eudoxus' theorems, perfecting many of Theaetetus', and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors. This man lived[2] in the time of the first Ptolemy. For Archimedes, who came immediately after the first (Ptolemy)[3], makes mention of Euclid: and, further, they say that Ptolemy once asked him if there was in geometry any shorter way than that of the elements, and he answered that there was no royal road to geometry[4]. He is then younger than the pupils of Plato but older than Eratosthenes and Archimedes; for the latter were contemporary with one another, as Eratosthenes somewhere says."

This passage shows that even Proclus had no direct knowledge of Euclid's birthplace or of the date of his birth or death. He proceeds by inference. Since Archimedes lived just after the first

[1] Proclus, ed. Friedlein, p. 68, 6—20.

[2] The word γέγονε must apparently mean "flourished," as Heiberg understands it (*Litterargeschichtliche Studien über Euklid*, 1882, p. 26), not "was born," as Hankel took it: otherwise part of Proclus' argument would lose its cogency.

[3] So Heiberg understands ἐπιβαλὼν τῷ πρώτῳ (sc. Πτολεμαίῳ). Friedlein's text has καὶ between ἐπιβαλὼν and τῷ πρώτῳ; and it is right to remark that another reading is καὶ ἐν τῷ πρώτῳ (without ἐπιβαλών) which has been translated "in his first *book*," by which is understood *On the Sphere and Cylinder* I., where (1) in Prop. 2 are the words "let *BC* be made equal to *D by the second* (proposition) *of the first* of Euclid's (books)," and (2) in Prop. 6 the words "For these things are handed down in the Elements" (without the name of Euclid). Heiberg thinks the former passage is referred to, and that Proclus must therefore have had before him the words "by the second of the first of Euclid": a fair proof that they are genuine, though in themselves they would be somewhat suspicious.

[4] The same story is told in Stobaeus, *Ecl.* (II. p. 228, 30, ed. Wachsmuth) about Alexander and Menaechmus. Alexander is represented as having asked Menaechmus to teach him geometry concisely, but he replied: "O king, through the country there are royal roads and roads for common citizens, but in geometry there is one road for all."

Ptolemy, and Archimedes mentions Euclid, while there is an anecdote about *some* Ptolemy and Euclid, *therefore* Euclid lived in the time of the first Ptolemy.

We may infer then from Proclus that Euclid was intermediate between the first pupils of Plato and Archimedes. Now Plato died in 347, Archimedes lived 287–212, Eratosthenes 276–194 B.C. Thus Euclid must have flourished *c.* 300 B.C., which date agrees well with the fact that Ptolemy reigned from 306 to 283 B.C.

It is most probable that Euclid received his mathematical training in Athens from the pupils of Plato; for most of the geometers who could have taught him were of that school, and it was in Athens that the older writers of elements, and the other mathematicians on whose works Euclid's *Elements* depend, had lived and taught. He may himself have been a Platonist, but this does not follow from the statements of Proclus on the subject. Proclus says namely that he was of the school of Plato and in close touch with that philosophy[1]. But this was only an attempt of a New Platonist to connect Euclid with his philosophy, as is clear from the next words in the same sentence, "for which reason also he set before himself, as the end of the whole Elements, the construction of the so-called Platonic figures." It is evident that it was only an idea of Proclus' own to infer that Euclid was a Platonist because his *Elements* end with the investigation of the five regular solids, since a later passage shows him hard put to it to reconcile the view that the construction of the five regular solids was the end and aim of the *Elements* with the obvious fact that they were intended to supply a foundation for the study of geometry in general, "to make perfect the understanding of the learner in regard to the whole of geometry[2]." To get out of the difficulty he says[3] that, if one should ask him what was the aim (σκοπός) of the treatise, he would reply by making a distinction between Euclid's intentions (1) as regards the subjects with which his investigations are concerned, (2) as regards the learner, and would say as regards (1) that "the whole of the geometer's argument is concerned with the cosmic figures." This latter statement is obviously incorrect. It is true that Euclid's *Elements* end with the construction of the five regular solids; but the planimetrical portion has no direct relation to them, and the arithmetical no relation at all; the propositions about them are merely the conclusion of the stereometrical division of the work.

One thing is however certain, namely that Euclid taught, and founded a school, at Alexandria. This is clear from the remark of Pappus about Apollonius[4]: "he spent a very long time with the pupils of Euclid at Alexandria, and it was thus that he acquired such a scientific habit of thought."

It is in the same passage that Pappus makes a remark which might, to an unwary reader, seem to throw some light on the

[1] Proclus, p. 68, 20, *καὶ τῇ προαιρέσει δὲ Πλατωνικός ἐστι καὶ τῇ φιλοσοφίᾳ ταύτῃ οἰκεῖος.*
[2] *ibid.* p. 71, 8. [3] *ibid.* p. 70, 19 sqq.
[4] Pappus, VII. p. 678, 10—12, *συσχολάσας τοῖς ὑπὸ Εὐκλείδου μαθηταῖς ἐν Ἀλεξανδρείᾳ πλεῖστον χρόνον, ὅθεν ἔσχε καὶ τὴν τοιαύτην ἕξιν οὐκ ἀμαθῆ.*

personality of Euclid. He is speaking about Apollonius' preface to the first book of his *Conics*, where he says that Euclid had not completely worked out the synthesis of the "three- and four-line locus," which in fact was not possible without some theorems first discovered by himself. Pappus says on this[1]: "Now Euclid—regarding Aristaeus as deserving credit for the discoveries he had already made in conics, and without anticipating him or wishing to construct anew the same system (such was his scrupulous fairness and his exemplary kindliness towards all who could advance mathematical science to however small an extent), being moreover in no wise contentious and, though exact, yet no braggart like the other [Apollonius]—wrote so much about the locus as was possible by means of the conics of Aristaeus, without claiming completeness for his demonstrations." It is however evident, when the passage is examined in its context, that Pappus is not following any tradition in giving this account of Euclid: he was offended by the terms of Apollonius' reference to Euclid, which seemed to him unjust, and he drew a fancy picture of Euclid in order to show Apollonius in a relatively unfavourable light.

Another story is told of Euclid which one would like to believe true. According to Stobaeus[2], "some one who had begun to read geometry with Euclid, when he had learnt the first theorem, asked Euclid, 'But what shall I get by learning these things?' Euclid called his slave and said 'Give him threepence, since he must make gain out of what he learns.'"

In the middle ages most translators and editors spoke of Euclid as Euclid *of Megara*. This description arose out of a confusion between our Euclid and the philosopher Euclid of Megara who lived about 400 B.C. The first trace of this confusion appears in Valerius Maximus (in the time of Tiberius) who says[3] that Plato, on being appealed to for a solution of the problem of doubling the cubical altar, sent the inquirers to "Euclid the geometer." There is no doubt about the reading, although an early commentator on Valerius Maximus wanted to correct "Eucliden" into "*Eudoxum*," and this correction is clearly right. But, if Valerius Maximus took Euclid the geometer for a contemporary of Plato, it could only be through confusing him with Euclid of Megara. The first specific reference to Euclid as Euclid of Megara belongs to the 14th century, occurring in the ὑπομνηματισμοί of Theodorus Metochita (d. 1332) who speaks of "Euclid of Megara, the Socratic philosopher, contemporary of Plato," as the author of treatises on plane and solid geometry, data, optics etc.: and a Paris MS. of the 14th century has "Euclidis philosophi Socratici liber elementorum." The misunderstanding was general in the period from Campanus' translation (Venice 1482) to those of Tartaglia (Venice 1565) and Candalla (Paris 1566). But one Constantinus Lascaris (d. about 1493) had already made the proper

[1] Pappus, VII. pp. 676, 25—678, 6. Hultsch, it is true, brackets the whole passage pp. 676, 25—678, 15, but apparently on the ground of the diction only.

[2] Stobaeus, *l.c.*

[3] VIII. 12, ext. 1.

distinction by saying of our Euclid that "he was different from him of Megara of whom Laertius wrote, and who wrote dialogues"[1]; and to Commandinus belongs the credit of being the first translator[2] to put the matter beyond doubt: "Let us then free a number of people from the error by which they have been induced to believe that our Euclid is the same as the philosopher of Megara" etc.

Another idea, that Euclid was born at Gela in Sicily, is due to the same confusion, being based on Diogenes Laertius' description[3] of the philosopher Euclid as being "of Megara, or, according to some, of Gela, as Alexander says in the *Διαδοχαί*."

In view of the poverty of Greek tradition on the subject even as early as the time of Proclus (410–485 A.D.), we must necessarily take *cum grano* the apparently circumstantial accounts of Euclid given by Arabian authors; and indeed the origin of their stories can be explained as the result (1) of the Arabian tendency to romance, and (2) of misunderstandings.

We read[4] that "Euclid, son of Naucrates, grandson of Zenarchus[5], called the author of geometry, a philosopher of somewhat ancient date, a Greek by nationality domiciled at Damascus, born at Tyre, most learned in the science of geometry, published a most excellent and most useful work entitled the foundation or elements of geometry, a subject in which no more general treatise existed before among the Greeks: nay, there was no one even of later date who did not walk in his footsteps and frankly profess his doctrine. Hence also Greek, Roman and Arabian geometers not a few, who undertook the task of illustrating this work, published commentaries, scholia, and notes upon it, and made an abridgment of the work itself. For this reason the Greek philosophers used to post up on the doors of their schools the well-known notice: 'Let no one come to our school, who has not first learned the elements of Euclid.'" The details at the beginning of this extract cannot be derived from Greek sources, for even Proclus did not know anything about Euclid's father, while it was not the Greek habit to record the names of grandfathers, as the Arabians commonly did. Damascus and Tyre were no doubt brought in to gratify a desire which the Arabians always showed to connect famous Greeks in some way or other with the East. Thus Naṣīraddīn, the translator of the *Elements*, who was of Ṭūs in Khurāsān, actually makes Euclid out to have been "Thusinus" also[6]. The readiness of the Arabians to run away with an idea is illustrated by the last words

[1] Letter to Fernandus Acuna, printed in Maurolycus, *Historia Siciliae*, fol. 21 r. (see Heiberg, *Euklid-Studien*, pp. 22—3, 25).

[2] Preface to translation (Pisauri, 1572).

[3] Diog. L. II. 106, p. 58 ed. Cobet.

[4] Casiri, *Bibliotheca Arabico-Hispana Escurialensis*, I. p. 339. Casiri's source is al-Qifṭī (d. 1248), the author of the *Ta'rīkh al-Ḥukamā*, a collection of biographies of philosophers, mathematicians, astronomers etc.

[5] The *Fihrist* says "son of Naucrates, the son of Berenice (?)" (see Suter's translation in *Abhandlungen zur Gesch. d. Math.* VI. Heft, 1892, p. 16).

[6] The same predilection made the Arabs describe Pythagoras as a pupil of the wise Salomo, Hipparchus as the exponent of Chaldaean philosophy or as the Chaldaean, Archimedes as an Egyptian etc. (Ḥājī Khalfa, *Lexicon Bibliographicum*, and Casiri).

of the extract. Everyone knows the story of Plato's inscription over the porch of the Academy: "let no one unversed in geometry enter my doors"; the Arab turned geometry into *Euclid's* geometry, and told the story of Greek philosophers in general and "*their* Academies."

Equally remarkable are the Arabian accounts of the relation of Euclid and Apollonius[1]. According to them the *Elements* were originally written, not by Euclid, but by a man whose name was Apollonius, a carpenter, who wrote the work in 15 books or sections[2]. In the course of time some of the work was lost and the rest became disarranged, so that one of the kings at Alexandria who desired to study geometry and to master this treatise in particular first questioned about it certain learned men who visited him and then sent for Euclid who was at that time famous as a geometer, and asked him to revise and complete the work and reduce it to order. Euclid then re-wrote it in 13 books which were thereafter known by his name. (According to another version Euclid composed the 13 books out of commentaries which he had published on two books of Apollonius on conics and out of introductory matter added to the doctrine of the five regular solids.) To the thirteen books were added two more books, the work of others (though some attribute these also to Euclid) which contain several things not mentioned by Apollonius. According to another version Hypsicles, a pupil of Euclid at Alexandria, offered to the king and published Books XIV. and XV., it being also stated that Hypsicles had "discovered" the books, by which it appears to be suggested that Hypsicles had edited them from materials left by Euclid.

We observe here the correct statement that Books XIV. and XV. were not written by Euclid, but along with it the incorrect information that Hypsicles, the author of Book XIV., wrote Book XV. also.

The whole of the fable about Apollonius having preceded Euclid and having written the *Elements* appears to have been evolved out of the preface to Book XIV. by Hypsicles, and in this way; the Book must in early times have been attributed to Euclid, and the inference based upon this assumption was left uncorrected afterwards when it was recognised that Hypsicles was the author. The preface is worth quoting:

"Basilides of Tyre, O Protarchus, when he came to Alexandria and met my father, spent the greater part of his sojourn with him on account of their common interest in mathematics. And once, when

[1] The authorities for these statements quoted by Casiri and Ḥājī Khalfa are al-Kindī's tract *de instituto libri Euclidis* (al-Kindī died about 873) and a commentary by Qāḍīzāde ar-Rūmī (d. about 1440) on a book called *Ashkāl at-ta' sīs* (fundamental propositions) by Ashraf Shamsaddīn as-Samarqandī (*c.* 1276) consisting of elucidations of 35 propositions selected from the first books of Euclid. Naṣīraddīn likewise says that Euclid cut out two of 15 books of elements then existing and published the rest under his own name. According to Qāḍīzāde the king heard that there was a celebrated geometer named Euclid at *Tyre*: Naṣīraddīn says that he sent for Euclid of Ṭūs.

[2] So says the *Fihrist*. Suter (*op. cit.* p. 49) thinks that the author of the *Fihrist* did not suppose Apollonius *of Perga* to be the writer of the *Elements*, as later Arabian authorities did, but that he distinguished another Apollonius whom he calls "a carpenter." Suter's argument is based on the fact that the *Fihrist's* article on Apollonius (of Perga) says nothing of the *Elements*, and that it gives the three great mathematicians, Euclid, Archimedes and Apollonius, in the correct chronological order.

examining the treatise written by Apollonius about the comparison between the dodecahedron and the icosahedron inscribed in the same sphere, (showing) what ratio they have to one another, they thought that Apollonius had not expounded this matter properly, and accordingly they emended the exposition, as I was able to learn from my father. And I myself, later, fell in with another book published by Apollonius, containing a demonstration relating to the subject, and I was greatly interested in the investigation of the problem. The book published by Apollonius is accessible to all—for it has a large circulation, having apparently been carefully written out later—but I decided to send you the comments which seem to me to be necessary, for you will through your proficiency in mathematics in general and in geometry in particular form an expert judgment on what I am about to say, and you will lend a kindly ear to my disquisition for the sake of your friendship to my father and your goodwill to me."

The idea that Apollonius preceded Euclid must evidently have been derived from the passage just quoted. It explains other things besides. Basilides must have been confused with βασιλεύς, and we have a probable explanation of the "Alexandrian king," and of the "learned men who visited" Alexandria. It is possible also that in the "Tyrian" of Hypsicles' preface we have the origin of the notion that Euclid was born in Tyre. These inferences argue, no doubt, very defective knowledge of Greek: but we could expect no better from those who took the *Organon* of Aristotle to be "instrumentum musicum pneumaticum," and who explained the name of Euclid, which they variously pronounced as *Uclides* or *Icludes*, to be compounded of *Ucli* a key, and *Dis* a measure, or, as some say, geometry, so that *Uclides* is equivalent to the *key of geometry*!

Lastly the alternative version, given in brackets above, which says that Euclid made the *Elements* out of commentaries which he wrote on two books of Apollonius on conics and prolegomena added to the doctrine of the five solids, seems to have arisen, through a like confusion, out of a later passage[1] in Hypsicles' Book XIV.: "And this is expounded by Aristaeus in the book entitled 'Comparison of the five figures,' and by Apollonius in the second edition of his comparison of the dodecahedron with the icosahedron." The "doctrine of the five solids" in the Arabic must be the "Comparison of the five figures" in the passage of Hypsicles, for nowhere else have we any information about a work bearing this title, nor can the Arabians have had. The reference to the *two books* of Apollonius on *conics* will then be the result of mixing up the fact that Apollonius wrote a book on conics with the *second edition* of the other work mentioned by Hypsicles. We do not find elsewhere in Arabian authors any mention of a commentary by Euclid on Apollonius and Aristaeus: so that the story in the passage quoted is really no more than a variation of the fable that the *Elements* were the work of Apollonius.

[1] Heiberg's Euclid, vol. v. p. 6.

CHAPTER II.

EUCLID'S OTHER WORKS.

IN giving a list of the Euclidean treatises other than the *Elements*, I shall be brief: for fuller accounts of them, or speculations with regard to them, reference should be made to the standard histories of mathematics[1].

I will take first the works which are mentioned by Greek authors.

1. The *Pseudaria*.

I mention this first because Proclus refers to it in the general remarks in praise of the *Elements* which he gives immediately after the mention of Euclid in his summary. He says[2]: "But, inasmuch as many things, while appearing to rest on truth and to follow from scientific principles, really tend to lead one astray from the principles and deceive the more superficial minds, he has handed down methods for the discriminative understanding of these things as well, by the use of which methods we shall be able to give beginners in this study practice in the discovery of paralogisms, and to avoid being misled. This treatise, by which he puts this machinery in our hands, he entitled (the book) of Pseudaria, enumerating in order their various kinds, exercising our intelligence in each case by theorems of all sorts, setting the true side by side with the false, and combining the refutation of error with practical illustration. This book then is by way of cathartic and exercise, while the Elements contain the irrefragable and complete guide to the actual scientific investigation of the subjects of geometry."

The book is considered to be irreparably lost. We may conclude however from the connexion of it with the *Elements* and the reference to its usefulness for beginners that it did not go outside the domain of elementary geometry[3].

[1] Heiberg gives very exhaustive details in his *Litterargeschichtliche Studien über Euklid*; the best of the shorter accounts are those of Cantor (*Gesch. d. Math.* I_3, 1907, pp. 278—294) and Loria (*Il periodo aureo della geometria greca*, p. 9 and pp. 63—85).

[2] Proclus, p. 70, 1—18.

[3] Heiberg points out that Alexander Aphrodisiensis appears to allude to the work in his commentary on Aristotle's *Sophistici Elenchi* (fol. 25 *b*): "Not only those (ἔλεγχοι) which do not start from the principles of the science, under which the problem is classed...but also those which do start from the proper principles of the science but in some respect admit a paralogism, e.g. the *Pseudographemata* of Euclid." Tannery (*Bull. des sciences math. et astr.* 2e Série, VI., 1882, 1ère Partie, p. 147) conjectures that it may be from this treatise that the same commentator got his information about the quadratures of the circle by Antiphon and

2. The *Data*.

The *Data* (δεδομένα) are included by Pappus in the *Treasury of Analysis* (τόπος ἀναλυόμενος), and he describes their contents[1]. They are still concerned with elementary geometry, though forming part of the introduction to higher analysis. Their form is that of propositions proving that, if certain things in a figure are given (in magnitude, in species, etc.), something else is given. The subject-matter is much the same as that of the planimetrical books of the *Elements*, to which the *Data* are often supplementary. We shall see this later when we come to compare the propositions in the *Elements* which give us the means of solving the general quadratic equation with the corresponding propositions of the *Data* which give the solution. The *Data* may in fact be regarded as elementary exercises in analysis.

It is not necessary to go more closely into the contents, as we have the full Greek text and the commentary by Marinus newly edited by Menge and therefore easily accessible[2].

3. The book *On divisions* (*of figures*).

This work (περὶ διαιρέσεων βιβλίον) is mentioned by Proclus[3]. In one place he is speaking of the conception or definition (λόγος) of *figure*, and of the divisibility of a figure into others differing from it in kind; and he adds: "For the circle is divisible into parts unlike in definition or notion (ἀνόμοια τῷ λόγῳ), and so is each of the rectilineal figures; this is in fact the business of the writer of the Elements in his Divisions, where he divides given figures, in one case into like figures, and in another into unlike[4]." "Like" and "unlike" here mean, not "similar" and "dissimilar" in the technical sense, but "like" or "unlike *in definition* or *notion*" (λόγῳ): thus to divide a triangle into triangles would be to divide it into "like" figures, to divide a triangle into a triangle and a quadrilateral would be to divide it into "unlike" figures.

The treatise is lost in Greek but has been discovered in the Arabic. First John Dee discovered a treatise *De divisionibus* by one Muḥammad Bagdadinus[5] and handed over a copy of it (in Latin) in 1563 to Commandinus, who published it, in Dee's name and his own, in 1570[6]. It was formerly supposed that Dee had himself translated

Bryson, to say nothing of the lunules of Hippocrates. I think however that there is an objection to this theory so far as regards Bryson; for Alexander distinctly says that Bryson's quadrature did *not* start from the proper principles of geometry, but from some principles more general.

[1] Pappus, VII. p. 638.

[2] Vol. VI. in the Teubner edition of *Euclidis opera omnia* by Heiberg and Menge. A translation of the *Data* is also included in Simson's Euclid (though naturally his text left much to be desired).

[3] Proclus, p. 69, 4.

[4] *ibid.* 144, 22—26.

[5] Steinschneider places him in the 10th c. H. Suter (*Bibliotheca Mathematica*, IV$_3$, 1903, pp. 24, 27) identifies him with Abū (Bekr) Muḥ. b. 'Abdalbāqī al-Baġdādī, Qāḍī (Judge) of Māristān (*circa* 1070–1141), to whom he also attributes the *Liber judei* (? judicis) *super decimum Euclidis* translated by Gherard of Cremona.

[6] *De superficierum divisionibus liber Machometo Bagdadino adscriptus, nunc primum Ioannis Dee Londinensis et Federici Commandini Urbinatis opera in lucem editus*, Pisauri, 1570, afterwards included in Gregory's Euclid (Oxford, 1703).

the tract into Latin from the Arabic[1]; but it now appears certain[2] that he found it in Latin in a Cotton MS. now in the British Museum. Dee, in his preface addressed to Commandinus, says nothing of his having *translated* the book, but only remarks that the very illegible MS. had caused him much trouble and (in a later passage) speaks of "the actual, very ancient, copy from which I *wrote out*..." (in ipso unde descripsi vetustissimo exemplari). The Latin translation of this tract from the Arabic was probably made by Gherard of Cremona (1114–1187), among the list of whose numerous translations a "liber divisionum" occurs. The Arabic original cannot have been a direct translation from Euclid, and probably was not even a direct adaptation of it; it contains mistakes and unmathematical expressions, and moreover does not contain the propositions about the division of a circle alluded to by Proclus. Hence it can scarcely have contained more than a fragment of Euclid's work.

But Woepcke found in a MS. at Paris a treatise in Arabic on the division of figures, which he translated and published in 1851[3]. It is expressly attributed to Euclid in the MS. and corresponds to the description of it by Proclus. Generally speaking, the divisions are divisions into figures of the same kind as the original figures, e.g. of triangles into triangles; but there are also divisions into "unlike" figures, e.g. that of a triangle by a straight line parallel to the base. The missing propositions about the division of a circle are also here: "to divide into two equal parts a given figure bounded by an arc of a circle and two straight lines including a given angle" and "to draw in a given circle two parallel straight lines cutting off a certain part of the circle." Unfortunately the proofs are given of only four propositions (including the two last mentioned) out of 36, because the Arabic translator found them too easy and omitted them. To illustrate the character of the problems dealt with I need only take one more example: "To cut off a certain fraction from a (parallel-) trapezium by a straight line which passes through a given point lying inside or outside the trapezium but so that a straight line can be drawn through it cutting both the parallel sides of the trapezium." The genuineness of the treatise edited by Woepcke is attested by the facts that the four proofs which remain are elegant and depend on propositions in the *Elements*, and that there is a lemma with a true Greek ring: "to apply to a straight line a rectangle equal to the rectangle contained by *AB*, *AC and deficient by a square*." Moreover the treatise is no fragment, but finishes with the words "end of the treatise," and is a well-ordered and compact whole. Hence we may safely conclude that Woepcke's is not only Euclid's own work but the whole of it[4]. A restoration of the work, with proofs, was attempted

[1] Heiberg, *Euklid-Studien*, p. 13.

[2] H. Suter in *Bibliotheca Mathematica*, IV_3, 1905–6, pp. 321—2.

[3] *Journal Asiatique*, 1851, p. 233 sqq.

[4] We are told by Casiri that Thābit b. Qurra emended the translation of the *liber de divisionibus*; but Ofterdinger seems to be wrong in saying that according to Gartz (*De interpretibus et explanatoribus Euclidis Arabicis schediasma historicum*, Halae, 1823) there is a

by Ofterdinger[1], who however does not give Woepcke's props. 30, 31, 34, 35, 36.

4. The *Porisms*.

It is not possible to give in this place any account of the controversies about the contents and significance of the three lost books of Porisms, or of the important attempts by Robert Simson and Chasles to restore the work. These may be said to form a whole literature, references to which will be found most abundantly given by Heiberg and Loria, the former of whom has treated the subject from the philological point of view, most exhaustively, while the latter, founding himself generally on Heiberg, has added useful details, from the mathematical side, relating to the attempted restorations, etc.[2] It must suffice here to give an extract from the only original source of information about the nature and contents of the *Porisms*, namely Pappus[3]. In his general preface about the books composing the *Treasury of Analysis* (τόπος ἀναλυόμενος) he says:

"After the Tangencies (of Apollonius) come, in three books, the Porisms of Euclid, [in the view of many] a collection most ingeniously devised for the analysis of the more weighty problems, [and] although nature presents an unlimited number of such porisms[4], [they have added nothing to what was written originally by Euclid, except that some before my time have shown their want of taste by adding to a few (of the propositions) second proofs, each (proposition) admitting of a definite number of demonstrations, as we have shown, and Euclid having given one for each, namely that which is the most lucid. These porisms embody a theory subtle, natural, necessary, and of considerable generality, which is fascinating to those who can see and produce results].

"Now all the varieties of porisms belong, neither to theorems nor problems, but to a species occupying a sort of intermediate position [so that their enunciations can be formed like those of either theorems or problems], the result being that, of the great number of geometers, some regarded them as of the class of theorems, and others of problems, looking only to the form of the proposition. But that the ancients knew better the difference between these three things, is clear from the definitions. For they said that a theorem is that which is proposed with a view to the demonstration of the very thing proposed, a problem that which is thrown out with a view to the construction of the very thing proposed, and a porism that which is proposed with a view to the producing of the very thing proposed. [But this definition of the porism was changed by the more recent writers who could not produce everything, but used these elements

complete MS. of Thābit's translation in the Escurial. I cannot find any such statement in Gartz.

[1] L. F. Ofterdinger, *Beiträge zur Wiederherstellung der Schrift des Euklides über die Theilung der Figuren*, Ulm, 1853.

[2] Heiberg, *Euklid-Studien*, pp. 56—79, and Loria, *Il periodo aureo della geometria greca*, pp. 70—82, 221—5.

[3] Pappus, ed. Hultsch, VII. pp. 648—660. I put in square brackets the words bracketed by Hultsch.

[4] I adopt Heiberg's reading of a comma here instead of a full stop.