13 Vector Analysis and Vector Fields

13.1 Basic Notions of the Theory of Vector Fields

13.1.1 Vector Functions of a Scalar Variable

13.1.1.1 Definitions

1. Vector Function of a Scalar Variable *t*

A vector function of a scalar variable is a vector $\vec{\mathbf{a}}$ whose components are real functions of t:

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}(t) = a_x(t)\vec{\mathbf{e}}_x + a_y(t)\vec{\mathbf{e}}_y + a_z(t)\vec{\mathbf{e}}_z. \tag{13.1}$$

The notions of limit, continuity, differentiability are defined componentwise for the vector $\vec{\mathbf{a}}(t)$.

2. Hodograph of a Vector Function

If we consider the vector function $\vec{\mathbf{a}}(t)$ as a position or radius vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}(t)$ of a point P, then this function describes a space curve while t varies **(Fig. 13.1)**. This space curve is called the *hodograph* of the vector function $\vec{\mathbf{a}}(t)$.

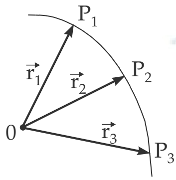

Figure 13.1

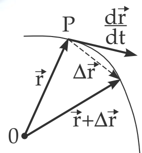

Figure 13.2

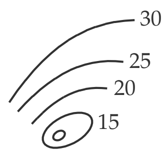

Figure 13.3

13.1.1.2 Derivative of a Vector Function

The derivative of (13.1) with respect to t is also a vector function of t:

$$\frac{d\vec{\mathbf{a}}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{a}}(t+\Delta t) - \vec{\mathbf{a}}(t)}{\Delta t} = \frac{da_x(t)}{dt}\vec{\mathbf{e}}_x + \frac{da_y(t)}{dt}\vec{\mathbf{e}}_y + \frac{da_z(t)}{dt}\vec{\mathbf{e}}_z. \tag{13.2}$$

The geometric representation of the derivative $\dfrac{d\vec{\mathbf{r}}}{dt}$ of the radius vector is a vector pointing in the direction of the tangent of the hodograph at the point P **(Fig. 13.2)**. Its length depends on the choice of the parameter t. If t is the time, then the vector $\vec{\mathbf{r}}(t)$ describes the motion of a point P in space (the space curve is its path), and $\dfrac{d\vec{\mathbf{r}}}{dt}$ has the direction and magnitude of the velocity of this motion. If $t = s$ is the arclength of this space curve, measured from a certain point, then obviously $\left|\dfrac{d\vec{\mathbf{r}}}{ds}\right| = 1$.

13.1.1.3 Rules of Differentiation for Vectors

$$\frac{d}{dt}\,(\vec{\mathbf{a}} \pm \vec{\mathbf{b}} \pm \vec{\mathbf{c}}) = \frac{d\vec{\mathbf{a}}}{dt} \pm \frac{d\vec{\mathbf{b}}}{dt} \pm \frac{d\vec{\mathbf{c}}}{dt}, \tag{13.3a}$$

$$\frac{d}{dt}\,(\varphi\vec{\mathbf{a}}) = \frac{d\varphi}{dt}\vec{\mathbf{a}} + \varphi\frac{d\vec{\mathbf{a}}}{dt} \qquad (\varphi \text{ is a scalar function of } t), \tag{13.3b}$$

$$\frac{d}{dt}\,(\vec{\mathbf{a}}\vec{\mathbf{b}}) = \frac{d\vec{\mathbf{a}}}{dt}\vec{\mathbf{b}} + \vec{\mathbf{a}}\frac{d\vec{\mathbf{b}}}{dt}, \tag{13.3c}$$

$$\frac{d}{dt}(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) = \frac{d\vec{\mathbf{a}}}{dt} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \frac{d\vec{\mathbf{b}}}{dt} \quad \text{(the factors must not be interchanged),} \tag{13.3d}$$

$$\frac{d}{dt}\vec{\mathbf{a}}[\varphi(t)] = \frac{d\vec{\mathbf{a}}}{d\varphi} \cdot \frac{d\varphi}{dt} \quad \text{(chain rule).} \tag{13.3e}$$

If $|\vec{\mathbf{a}}(t)| = \text{const}$, i.e., $\vec{\mathbf{a}}^2(t) = \vec{\mathbf{a}}(t) \cdot \vec{\mathbf{a}}(t) = \text{const}$, then it follows from (13.3c) that $\vec{\mathbf{a}} \cdot \frac{d\vec{\mathbf{a}}}{dt} = 0$, i.e., $\frac{d\vec{\mathbf{a}}}{dt}$ and $\vec{\mathbf{a}}$ are perpendicular to each other. Examples of this fact:

■ **A:** Radius and tangent vectors of a circle in the plane and

■ **B:** position and tangent vectors of a curve on the sphere. Then the hodograph is a *spherical curve*.

13.1.1.4 Taylor Expansion for Vector Functions

$$\vec{\mathbf{a}}(t+h) = \vec{\mathbf{a}}(t) + h\frac{d\vec{\mathbf{a}}}{dt} + \frac{h^2}{2!}\frac{d^2\vec{\mathbf{a}}}{dt^2} + \cdots + \frac{h^n}{n!}\frac{d^n\vec{\mathbf{a}}}{dt^n} + \cdots. \tag{13.4}$$

The expansion of a vector function in a Taylor series makes sense only if it is convergent. Because the limit is defined componentwise, the convergence can be checked componentwise, so the convergence of this series with vector terms can be determined exactly by the same methods as the convergence of a series with complex terms (see 14.3.2, p. 691). So the convergence of a series with vector terms is reduced to the convergence of a series with scalar terms.
The differential of a vector function $\vec{\mathbf{a}}(t)$ is defined by:

$$d\vec{\mathbf{a}} = \frac{d\vec{\mathbf{a}}}{dt}\Delta t. \tag{13.5}$$

13.1.2 Scalar Fields

13.1.2.1 Scalar Field or Scalar Point Function

If we assign a number (scalar value) U to every point P of a subset of space, then we write

$$U = U(P) \tag{13.6a}$$

and we call (13.6a) a *scalar field* (or *scalar function*).

■ Examples of scalar fields are temperature, density, potential, etc., of solids.

A scalar field $U = U(P)$ can also be considered as

$$U = U(\vec{\mathbf{r}}), \tag{13.6b}$$

where $\vec{\mathbf{r}}$ is the position vector of the point P with a given pole 0 (see 3.5.1.1, **6.**, p. 181).

13.1.2.2 Important Special Cases of Scalar Fields

1. Plane Field

We have a plane field, if the function is defined only for the points of a plane in space.

2. Central Field

If a function has the same value at all points P lying at the same distance from a fixed point $C(\vec{\mathbf{r}}_1)$, called the center, then we call it a *central symmetric field* or also a *central or spherical field*. The function U depends only on the distance $\overline{CP} = |\vec{\mathbf{r}}|$:

$$U = f(|\vec{\mathbf{r}}|). \tag{13.7a}$$

■ The field of the intensity of a point-like source, e.g., the field of brightness of a point-like source of light at the pole, can be described with $|\vec{\mathbf{r}}| = r$ as the distance from the light source:

$$U = \frac{c}{r^2} \quad (c \text{ const}). \tag{13.7b}$$

3. Axial Field

If the function U has the same value at all points lying at an equal distance from a certain straight line (axis of the field) then the field is called *cylindrically symmetric* or an *axially symmetric field*, or briefly an *axial field*.

13.1.2.3 Coordinate Definition of a Field

If the points of a subset of space are given by their coordinates, e.g., by Cartesian, cylindrical, or spherical coordinates, then the corresponding scalar field (13.6a) is represented, in general, by a function of three variables:

$$U = \Phi(x, y, z), \qquad U = \Psi(\rho, \varphi, z) \qquad \text{or} \qquad U = \chi(r, \vartheta, \varphi). \tag{13.8a}$$

In the case of a plane field, a function with two variables is sufficient. It has the form in Cartesian and polar coordinates:

$$U = \Phi(x, y) \qquad \text{or} \qquad U = \Psi(\rho, \varphi). \tag{13.8b}$$

The functions in (13.8a) and (13.8b), in general, are assumed to be continuous, except, maybe, at some points, curves or surfaces of discontinuity. The functions have the form

a) for a central field: $$U = U(\sqrt{x^2 + y^2 + z^2}) = U(\sqrt{\rho^2 + z^2}) = U(r), \tag{13.9a}$$

b) for an axial field: $$U = U(\sqrt{x^2 + y^2}) = U(\rho) = U(r \sin\vartheta). \tag{13.9b}$$

Dealing with central fields is easiest using spherical coordinates, with axial fields using cylindrical coordinates.

13.1.2.4 Level Surfaces and Level Lines of a Field

1. Level Surface

A level surface is the union of all points in space where the function (13.6a) has a constant value

$$U = \text{const.} \tag{13.10a}$$

Different constants $U_0, U_1, U_2, \ldots$ define different level surfaces. There is a level surface passing through every point except the points where the function is not defined. The level surface equations in the three coordinate systems used so far are:

$$U = \Phi(x, y, z) = \text{const}, \qquad U = \Psi(\rho, \varphi, z) = \text{const}, \qquad U = \chi(r, \vartheta, \varphi) = \text{const.} \tag{13.10b}$$

■ Examples of level surfaces of different fields:

A: $U = \vec{\mathbf{c}}\,\vec{\mathbf{r}} = c_x x + c_y y + c_z z$:	Parallel planes.
B: $U = x^2 + 2y^2 + 4z^2$:	Similar ellipsoids in similar positions.
C: Central field:	Concentric spheres.
D: Axial field:	Coaxial cylinders.

2. Level Lines

Level lines replace level surfaces in plane fields. They satisfy the equation

$$U = \text{const.} \tag{13.11}$$

Level lines are usually drawn for equal intervals of U and each of them is marked by the corresponding value of U (**Fig. 13.3**).

■ Well-known examples are the isobaric lines on a synoptic map or the contour lines on topographic maps.

In particular cases, level surfaces degenerate into points or lines, and level lines degenerate into separate points.

■ The level lines of the fields a) $U = xy$, b) $U = \dfrac{y}{x^2}$, c) $U = r^2$, d) $U = \dfrac{1}{r}$ are represented in **Fig. 13.4**.

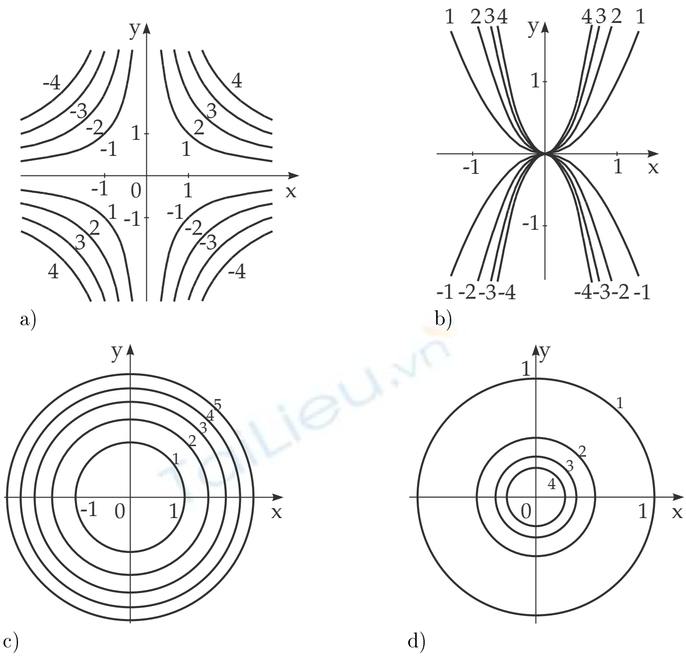

Figure 13.4

13.1.3 Vector Fields

13.1.3.1 Vector Field or Vector Point Function

If we assign a vector $\vec{\mathbf{V}}$ to every point P of a subset of space, then we denote it by

$$\vec{\mathbf{V}} = \vec{\mathbf{V}}(P) \tag{13.12a}$$

and we call (13.12a) a *vector field*.

■ Examples of vector fields are the velocity field of a fluid in motion, a field of force, and a magnetic or electric intensity field.

A vector field $\vec{\mathbf{V}} = \vec{\mathbf{V}}(P)$ can be regarded as a vector function

$$\vec{\mathbf{V}} = \vec{\mathbf{V}}(\vec{\mathbf{r}}), \tag{13.12b}$$

where $\vec{\mathbf{r}}$ is the position vector of the point P with a given pole 0. If all values of $\vec{\mathbf{r}}$ as well as $\vec{\mathbf{V}}$ lie in a plane, then the field is called a plane vector field (see 3.5.2, p. 189).

13.1.3.2 Important Cases of Vector Fields

1. Central Vector Field

In a central vector field all vectors $\vec{\mathbf{V}}$ lie on straight lines passing through a fixed point called the *center* **(Fig. 13.5a)**.

If we locate the pole at the center, then the field is defined by the formula

$$\vec{\mathbf{V}} = f(\vec{\mathbf{r}})\,\vec{\mathbf{r}}, \tag{13.13a}$$

where all the vectors have the same direction as the radius vector $\vec{\mathbf{r}}$. It often has some advantage to define the field by the formula

$$\vec{\mathbf{V}} = \varphi(\vec{\mathbf{r}})\frac{\vec{\mathbf{r}}}{r}, \tag{13.13b}$$

where $\varphi(\vec{\mathbf{r}})$ is the length of the vector $\vec{\mathbf{V}}$ and $\dfrac{\vec{\mathbf{r}}}{r}$ is a unit vector.

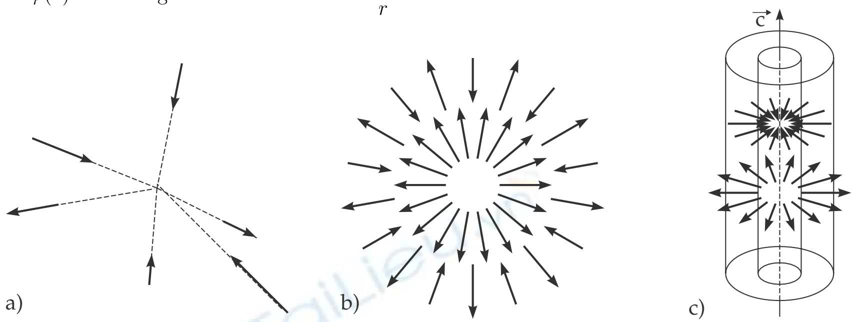

Figure 13.5

2. Spherical Vector Field

A spherical vector field is a special case of a central vector field, where the length of the vector $\vec{\mathbf{V}}$ depends only on the distance $|\vec{\mathbf{r}}|$ **(Fig. 13.5b)**.

■ Examples are the *Newton* and the *Coulomb force field* of a point-like mass or of a point-like electric charge:

$$\vec{\mathbf{V}} = \frac{c}{r^3}\vec{\mathbf{r}} = \frac{c}{r^2}\frac{\vec{\mathbf{r}}}{r} \quad (c \text{ const})\,. \tag{13.14}$$

The special case of a plane spherical vector field is called a *circular field*.

3. Cylindrical Vector Field

a) All vectors $\vec{\mathbf{V}}$ lie on straight lines intersecting a certain line (called the *axis*) and perpendicular to it, and

b) all vectors $\vec{\mathbf{V}}$ at the points lying at the same distance from the axis have equal length, and they are directed either toward the axis or away from it **(Fig. 13.5c)**.

If we locate the pole on the axis parallel to the unit vector $\vec{\mathbf{c}}$, then the field has the form

$$\vec{\mathbf{V}} = \varphi(\rho)\frac{\vec{\mathbf{r}}^*}{\rho}, \tag{13.15a}$$

where $\vec{\mathbf{r}}^*$ is the projection of $\vec{\mathbf{r}}$ on a plane perpendicular to the axis:

$$\vec{\mathbf{r}}^* = \vec{\mathbf{c}} \times (\vec{\mathbf{r}} \times \vec{\mathbf{c}}). \tag{13.15b}$$

By intersecting this field with planes perpendicular to the axis, we always get equal circular fields.

13.1.3.3 Coordinate Representation of Vector Fields

1. Vector Field in Cartesian Coordinates

The vector field (13.12a) can be defined by scalar fields $V_1(\vec{\mathbf{r}})$, $V_2(\vec{\mathbf{r}})$, and $V_3(\vec{\mathbf{r}})$ which are the coordinate functions of $\vec{\mathbf{V}}$, i.e., the coefficients of its decomposition into any three non-coplanar base vectors $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$, and $\vec{\mathbf{e}}_3$:

$$\vec{\mathbf{V}} = V_1\vec{\mathbf{e}}_1 + V_2\vec{\mathbf{e}}_2 + V_3\vec{\mathbf{e}}_3. \tag{13.16a}$$

If we take the coordinate unit vectors $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, and $\vec{\mathbf{k}}$ as the base vectors and express the coefficients V_1, V_2, V_3 in Cartesian coordinates, then we get:

$$\vec{\mathbf{V}} = V_x(x,y,z)\vec{\mathbf{i}} + V_y(x,y,z)\vec{\mathbf{j}} + V_z(x,y,z)\vec{\mathbf{k}}. \tag{13.16b}$$

So, the vector field can be defined with the help of three scalar functions of three scalar variables.

2. Vector Field in Cylindrical and Spherical Coordinates

In cylindrical and spherical coordinates, the coordinate unit vectors

$$\vec{\mathbf{e}}_\rho,\ \vec{\mathbf{e}}_\varphi,\ \vec{\mathbf{e}}_z\ (=\vec{\mathbf{k}}), \qquad \text{and} \qquad \vec{\mathbf{e}}_r\ (=\frac{\vec{\mathbf{r}}}{r}),\ \vec{\mathbf{e}}_\vartheta,\ \vec{\mathbf{e}}_\varphi \tag{13.17a}$$

are tangents to the coordinate lines at each point **(Fig. 13.6, 13.7)**. In this order they always form a right-handed system. The coefficients are expressed as functions of the corresponding coordinates:

$$\vec{\mathbf{V}} = V_\rho(\rho,\varphi,z)\vec{\mathbf{e}}_\rho + V_\varphi(\rho,\varphi,z)\vec{\mathbf{e}}_\varphi + V_z(\rho,\varphi,z)\vec{\mathbf{e}}_z\,, \tag{13.17b}$$

$$\vec{\mathbf{V}} = V_r(r,\vartheta,\varphi)\vec{\mathbf{e}}_r + V_\vartheta(r,\vartheta,\varphi)\vec{\mathbf{e}}_\vartheta + V_\varphi(r,\vartheta,\varphi)\vec{\mathbf{e}}_\varphi. \tag{13.17c}$$

At transition from one point to the other, the coordinate unit vectors change their directions, but remain mutually perpendicular.

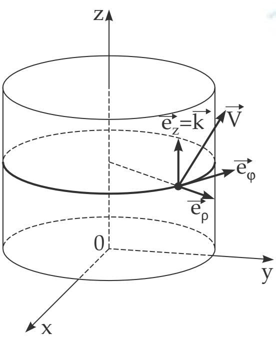

Figure 13.6

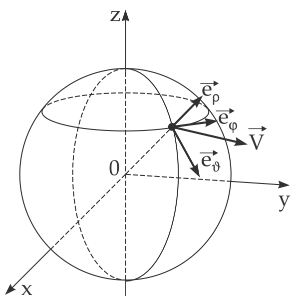

Figure 13.7

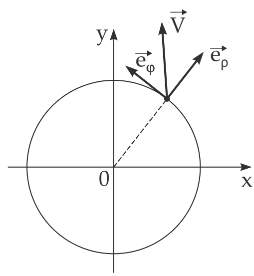

Figure 13.8

13.1.3.4 Transformation of Coordinate Systems

See also **Table 13.1**.

1. Cartesian Coordinates in Terms of Cylindrical Coordinates

$$V_x = V_\rho\cos\varphi - V_\varphi\sin\varphi, \qquad V_y = V_\rho\sin\varphi + V_\varphi\cos\varphi, \qquad V_z = V_z. \tag{13.18}$$

2. Cylindrical Coordinates in Terms of Cartesian Coordinates

$$V_\rho = V_x\cos\varphi + V_y\sin\varphi, \qquad V_\varphi = -V_x\sin\varphi + V_y\cos\varphi, \qquad V_z = V_z. \tag{13.19}$$

3. Cartesian Coordinates in Terms of Spherical Coordinates

$$\begin{aligned} V_x &= V_r\sin\vartheta\cos\varphi - V_\varphi\sin\varphi + V_\vartheta\cos\varphi\cos\vartheta, \\ V_y &= V_r\sin\vartheta\sin\varphi + V_\varphi\cos\varphi + V_\vartheta\sin\varphi\cos\vartheta, \\ V_z &= V_r\cos\vartheta - V_\vartheta\sin\vartheta. \end{aligned} \tag{13.20}$$

4. Spherical Coordinates in Terms of Cartesian Coordinates

$$\begin{aligned} V_r &= V_x\sin\vartheta\cos\varphi + V_y\sin\vartheta\sin\varphi + V_z\cos\vartheta, \\ V_\vartheta &= V_x\cos\vartheta\cos\varphi + V_y\cos\vartheta\sin\varphi - V_z\sin\vartheta, \end{aligned} \tag{13.21}$$

$$V_\varphi = -V_x \sin\varphi + V_y \cos\varphi.$$

5. Expression of a Spherical Vector Field in Cartesian Coordinates

$$\vec{\mathbf{V}} = \varphi(\sqrt{x^2+y^2+z^2})(x\vec{\mathbf{i}} + y\vec{\mathbf{j}} + z\vec{\mathbf{k}}). \tag{13.22}$$

6. Expression of a Cylindrical Vector Field in Cartesian Coordinates

$$\vec{\mathbf{V}} = \varphi(\sqrt{x^2+y^2})(x\vec{\mathbf{i}} + y\vec{\mathbf{j}}). \tag{13.23}$$

In the case of a spherical vector field, spherical coordinates are most convenient for investigations, i.e., the form $\vec{\mathbf{V}} = V(r)\vec{\mathbf{e}}_r$; and for investigations in cylindrical fields, cylindrical coordinates are most convenient, i.e., the form $\vec{\mathbf{V}} = V(\varphi)\vec{\mathbf{e}}_\varphi$. In the case of a plane field **(Fig. 13.8)**, we have

$$\vec{\mathbf{V}} = V_x(x,y)\vec{\mathbf{i}} + V_y(x,y)\vec{\mathbf{j}} = V_\rho(\rho,\varphi)\vec{\mathbf{e}}_\rho + V_\varphi(\rho,\varphi)\vec{\mathbf{e}}_\varphi, \tag{13.24}$$

and for a circular field

$$\vec{\mathbf{V}} = \varphi(\sqrt{x^2+y^2})(x\vec{\mathbf{i}} + y\vec{\mathbf{j}}) = \varphi(\rho)\vec{\mathbf{e}}_\rho. \tag{13.25}$$

Table 13.1 Relations between the components of a vector in Cartesian, cylindrical, and spherical coordinates

Cartesian coordinates	**Cylindrical coord.**	**Spherical coordinates**
$\vec{\mathbf{V}} = V_x\vec{\mathbf{e}}_x + V_y\vec{\mathbf{e}}_y + V_z\vec{\mathbf{e}}_z$	$V_\rho\vec{\mathbf{e}}_\rho + V_\varphi\vec{\mathbf{e}}_\varphi + V_z\vec{\mathbf{e}}_z$	$V_r\vec{\mathbf{e}}_r + V_\vartheta\vec{\mathbf{e}}_\vartheta + V_\varphi\vec{\mathbf{e}}_\varphi$
V_x	$= V_\rho\cos\varphi - V_\varphi\sin\varphi$	$= V_r\sin\vartheta\cos\varphi + V_\vartheta\cos\vartheta\cos\varphi - V_\varphi\sin\varphi$
V_y	$= V_\rho\sin\varphi + V_\varphi\cos\varphi$	$= V_r\sin\vartheta\sin\varphi + V_\vartheta\cos\vartheta\sin\varphi + V_\varphi\cos\varphi$
V_z	$= V_z$	$= V_r\cos\vartheta - V_\vartheta\sin\vartheta$
$V_x\cos\varphi + V_y\sin\varphi$	$= V_\rho$	$= V_r\sin\vartheta + V_\vartheta\cos\vartheta$
$-V_x\sin\varphi + V_y\cos\varphi$	$= V_\varphi$	$= V_\varphi$
V_z	$= V_z$	$= V_r\cos\vartheta - V_\vartheta\sin\vartheta$
$V_x\sin\vartheta\cos\varphi + V_y\sin\vartheta\sin\varphi + V_z\cos\vartheta$	$= V_\rho\sin\vartheta + V_z\cos\vartheta$	$= V_r$
$V_x\cos\vartheta\cos\varphi + V_y\cos\vartheta\sin\varphi - V_z\sin\vartheta$	$= V_\rho\cos\vartheta - V_z\sin\vartheta$	$= V_\vartheta$
$-V_x\sin\varphi + V_y\cos\varphi$	$= V_\varphi$	$= V_\varphi$

13.1.3.5 Vector Lines

A curve C is called a *line of a vector* or a *vector line* of the vector field $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ **(Fig. 13.9)** if the vector $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ is a tangent vector of the curve at every point P. There is a vector line passing through every point of the field. Vector lines do not intersect each other, except, maybe, at points where the function $\vec{\mathbf{V}}$ is not defined, or where it is the zero vector. The differential equations of the vector lines of a vector field $\vec{\mathbf{V}}$ given in Cartesian coordinates are

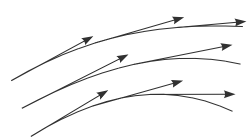

Figure 13.9

a) in general: $\dfrac{dx}{V_x} = \dfrac{dy}{V_y} = \dfrac{dz}{V_z}$, (13.26a) **b) for a plane field:** $\dfrac{dx}{V_x} = \dfrac{dy}{V_y}$. (13.26b)

To solve these differential equations see 9.1.1.2, p. 489 or 9.2.1.1, p. 517.

■ **A:** The vector lines of a central field are rays starting at the center of the vector field.

■ **B:** The vector lines of the vector field $\vec{\mathbf{V}} = \vec{\mathbf{c}} \times \vec{\mathbf{r}}$ are circles lying in planes perpendicular to the vector $\vec{\mathbf{c}}$. Their centers are on the axis parallel to $\vec{\mathbf{c}}$.

13.2 Differential Operators of Space

13.2.1 Directional and Space Derivatives

13.2.1.1 Directional Derivative of a Scalar Field

The directional derivative of a scalar field $U = U(\vec{\mathbf{r}})$ at a point P with position vector $\vec{\mathbf{r}}$ in the direction $\vec{\mathbf{c}}$ **(Fig. 13.10)** is defined as the limit of the quotient

$$\frac{\partial U}{\partial \vec{\mathbf{c}}} = \lim_{\varepsilon \to 0} \frac{U(\vec{\mathbf{r}} + \varepsilon \vec{\mathbf{c}}) - U(\vec{\mathbf{r}})}{\varepsilon}. \tag{13.27}$$

If the derivative of the field $U = U(\vec{\mathbf{r}})$ at a point $\vec{\mathbf{r}}$ in the direction of the unit vector $\vec{\mathbf{c}}^0$ of $\vec{\mathbf{c}}$ is denoted by $\dfrac{\partial U}{\partial \vec{\mathbf{c}}^0}$, then the relation between the derivative of the function with respect to the vector $\vec{\mathbf{c}}$ and with respect to its unit vector $\vec{\mathbf{c}}^0$ at the same point is

$$\frac{\partial U}{\partial \vec{\mathbf{c}}} = |\vec{\mathbf{c}}| \frac{\partial U}{\partial \vec{\mathbf{c}}^0}. \tag{13.28}$$

The derivative $\dfrac{\partial U}{\partial \vec{\mathbf{c}}^0}$ with respect to the unit vector represents the speed of increase of the function U in the direction of the vector $\vec{\mathbf{c}}^0$ at the point $\vec{\mathbf{r}}$. If $\vec{\mathbf{n}}$ is the normal unit vector to the level surface passing through the point $\vec{\mathbf{r}}$, and $\vec{\mathbf{n}}$ is pointing in the direction of increasing U, then $\dfrac{\partial U}{\partial \vec{\mathbf{n}}}$ has the greatest value among all the derivatives at the point with respect to the unit vectors in different directions. Between the directional derivatives with respect to $\vec{\mathbf{n}}$ and with respect to any direction $\vec{\mathbf{c}}^0$, we have the relation

$$\frac{\partial U}{\partial \vec{\mathbf{c}}^0} = \frac{\partial U}{\partial \vec{\mathbf{n}}} \cos(\vec{\mathbf{c}}^0, \vec{\mathbf{n}}) = \frac{\partial U}{\partial \vec{\mathbf{n}}} \cos\varphi = \vec{\mathbf{c}}^0 \cdot \operatorname{grad} U \quad \text{(see (13.35), p. 651)}. \tag{13.29}$$

In the following, directional derivatives always mean the directional derivative with respect to a unit vector.

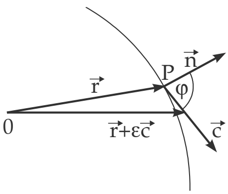

Figure 13.10

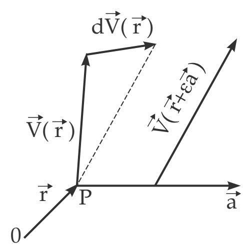

Figure 13.11

13.2.1.2 Directional Derivative of a Vector Field

The directional derivative of a vector field is defined analogously to the directional derivative of a scalar field. The directional derivative of the vector field $\vec{\mathbf{V}} = \vec{\mathbf{V}}(\vec{\mathbf{r}})$ at a point P with position vector $\vec{\mathbf{r}}$ **(Fig. 13.11)** with respect to the vector $\vec{\mathbf{a}}$ is defined as the limit of the quotient

$$\frac{\partial \vec{\mathbf{V}}}{\partial \vec{\mathbf{a}}} = \lim_{\varepsilon \to 0} \frac{\vec{\mathbf{V}}(\vec{\mathbf{r}} + \varepsilon \vec{\mathbf{a}}) - \vec{\mathbf{V}}(\vec{\mathbf{r}})}{\varepsilon}. \tag{13.30}$$

If the derivative of the vector field $\vec{\mathbf{V}} = \vec{\mathbf{V}}(\vec{\mathbf{r}})$ at a point $\vec{\mathbf{r}}$ in the direction of the unit vector $\vec{\mathbf{a}}^0$ of $\vec{\mathbf{a}}$ is denoted by $\dfrac{\partial \vec{\mathbf{V}}}{\partial \vec{\mathbf{a}}^0}$, then

$$\frac{\partial \vec{\mathbf{V}}}{\partial \vec{\mathbf{a}}} = |\vec{\mathbf{a}}| \frac{\partial \vec{\mathbf{V}}}{\partial \vec{\mathbf{a}}^0}. \tag{13.31}$$

In Cartesian coordinates, i.e., for $\vec{\mathbf{V}} = V_x \vec{\mathbf{e}}_x + V_y \vec{\mathbf{e}}_y + V_z \vec{\mathbf{e}}_z$, $\vec{\mathbf{a}} = a_x \vec{\mathbf{e}}_x + a_y \vec{\mathbf{e}}_y + a_z \vec{\mathbf{e}}_z$, we have:

$$\frac{\partial \vec{\mathbf{V}}}{\partial \vec{\mathbf{a}}} = (\vec{\mathbf{a}} \cdot \operatorname{grad})\vec{\mathbf{V}} = (\vec{\mathbf{a}} \cdot \operatorname{grad} V_x)\vec{\mathbf{e}}_{\mathbf{x}} + (\vec{\mathbf{a}} \cdot \operatorname{grad} V_y)\vec{\mathbf{e}}_{\mathbf{y}} + (\vec{\mathbf{a}} \cdot \operatorname{grad} V_z)\vec{\mathbf{e}}_{\mathbf{z}}. \tag{13.32a}$$

In general coordinates we have:

$$\begin{aligned} \frac{\partial \vec{\mathbf{V}}}{\partial \vec{\mathbf{a}}} &= (\vec{\mathbf{a}} \cdot \operatorname{grad})\vec{\mathbf{V}} \\ &= \frac{1}{2}(\operatorname{rot}(\vec{\mathbf{V}} \times \vec{\mathbf{a}}) + \operatorname{grad}(\vec{\mathbf{a}} \cdot \vec{\mathbf{V}}) + \vec{\mathbf{a}} \operatorname{div} \vec{\mathbf{V}} - \vec{\mathbf{V}} \operatorname{div} \vec{\mathbf{a}} - \vec{\mathbf{a}} \times \operatorname{rot} \vec{\mathbf{V}} - \vec{\mathbf{V}} \times \operatorname{rot} \vec{\mathbf{a}}. \end{aligned} \tag{13.32b}$$

13.2.1.3 Volume Derivative

Volume derivatives of a scalar field $U = U(\vec{\mathbf{r}})$ or a vector field $\vec{\mathbf{V}}$ at a point $\vec{\mathbf{r}}$ are quantities of three forms, which are obtained as follows:

1. We surround the point $\vec{\mathbf{r}}$ of the scalar field or of the vector field by a closed surface Σ. This surface can be represented in parametric form $\vec{\mathbf{r}} = \vec{\mathbf{r}}(u, v) = x(u, v)\vec{\mathbf{e}}_x + y(u, v)\vec{\mathbf{e}}_y + z(u, v)\vec{\mathbf{e}}_z$, so the corresponding vectorial surface element is

$$d\vec{\mathbf{S}} = \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v}\, du\, dv. \tag{13.33a}$$

2. We evaluate the surface integral over the closed surface Σ. Here, the following three types of integrals can be considered:

$$\oint\!\!\!\oint_{(\Sigma)} U\, d\vec{\mathbf{S}}, \qquad \oint\!\!\!\oint_{(\Sigma)} \vec{\mathbf{V}} \cdot d\vec{\mathbf{S}}, \qquad \oint\!\!\!\oint_{(\Sigma)} \vec{\mathbf{V}} \times d\vec{\mathbf{S}}. \tag{13.33b}$$

3. We determine the limits (if they exist)

$$\lim_{V \to 0} \frac{1}{V} \oint\!\!\!\oint_{(\Sigma)} U\, d\vec{\mathbf{S}}, \quad \lim_{V \to 0} \frac{1}{V} \oint\!\!\!\oint_{(\Sigma)} \vec{\mathbf{V}} \cdot d\vec{\mathbf{S}}, \quad \lim_{V \to 0} \frac{1}{V} \oint\!\!\!\oint_{(\Sigma)} \vec{\mathbf{V}} \times d\vec{\mathbf{S}}. \tag{13.33c}$$

Here V denotes the volume of the region of space that contains the point with the position vector $\vec{\mathbf{r}}$ inside, and which is bounded by the considered closed surface Σ.

The limits (13.33c) are called volume derivatives. The *gradient of a scalar field* and the *divergence* and the *rotation* of a vector field can be derived from them in the given order. In the following paragraphs, we discuss these notions in details (we will even define them again.)

13.2.2 Gradient of a Scalar Field

The gradient of a scalar field can be defined in different ways.

13.2.2.1 Definition of the Gradient

The *gradient* of a function U is a vector grad U, which can be assigned to every point of a scalar field $U = U(\vec{\mathbf{r}})$, having the following properties:

1. The direction of grad U is always perpendicular to the direction of the level surface U = const, passing through the considered point,

2. grad U is always in the direction in which the function U is increasing,

3. $|\operatorname{grad} U| = \dfrac{\partial U}{\partial \vec{\mathbf{n}}}$, i.e., the magnitude of grad U is equal to the directional derivative of U in the *normal direction*.

If the gradient is defined in another way, e.g., as a volume derivative or by the differential operator, then the previous defining properties became consequences of the definition.

13.2.2.2 Gradient and Volume Derivative

The *gradient* U of the scalar field $U = U(\vec{\mathbf{r}})$ at a point $\vec{\mathbf{r}}$ can be defined as its *volume derivative*. If the following limit exists, then we call it the *gradient of* U at $\vec{\mathbf{r}}$:

$$\operatorname{grad} U = \lim_{V \to 0} \frac{\oiint\limits_{(\Sigma)} U \, d\vec{\mathbf{S}}}{V} . \tag{13.34}$$

Here V is the volume of the region of space containing the point belonging to $\vec{\mathbf{r}}$ inside and bounded by the closed surface Σ. (If the independent variable is not a three-dimensional vector, then the gradient is defined by the differential operator.)

13.2.2.3 Gradient and Directional Derivative

The directional derivative of the scalar field U with respect to the unit vector $\vec{\mathbf{c}}^{\,0}$ is equal to the projection of grad U onto the direction of the unit vector $\vec{\mathbf{c}}^{\,0}$:

$$\frac{\partial U}{\partial \vec{\mathbf{c}}^{\,0}} = \vec{\mathbf{c}}^{\,0} \cdot \operatorname{grad} U, \tag{13.35}$$

i.e., the directional derivative can be calculated as the dot product of the gradient and the unit vector pointing into the required direction.

Remark: The directional derivative at certain points in certain directions may also exist if the gradient does not exist there.

13.2.2.4 Further Properties of the Gradient

1. The absolute value of the gradient is greater if the level lines or level surfaces drawn as mentioned in 13.1.2.4, **2.**, p. 644, are more dense.

2. The gradient is the zero vector ($\operatorname{grad} U = \vec{\mathbf{0}}$) if U has a maximum or minimum at the considered point. The level lines or surfaces degenerate to a point there.

13.2.2.5 Gradient of the Scalar Field in Different Coordinates

1. Gradient in Cartesian Coordinates

$$\operatorname{grad} U = \frac{\partial U(x,y,z)}{\partial x}\vec{\mathbf{i}} + \frac{\partial U(x,y,z)}{\partial y}\vec{\mathbf{j}} + \frac{\partial U(x,y,z)}{\partial z}\vec{\mathbf{k}}. \tag{13.36}$$

2. Gradient in Cylindrical Coordinates ($x = \rho\cos\varphi,\ y = \rho\sin\varphi,\ z = z$)

$$\operatorname{grad} U = \operatorname{grad}_\rho U \vec{\mathbf{e}}_\rho + \operatorname{grad}_\varphi U \vec{\mathbf{e}}_\varphi + \operatorname{grad}_z U \vec{\mathbf{e}}_z \quad \text{with} \tag{13.37a}$$

$$\operatorname{grad}_\rho U = \frac{\partial U}{\partial \rho}, \quad \operatorname{grad}_\varphi U = \frac{1}{\rho}\frac{\partial U}{\partial \varphi}, \quad \operatorname{grad}_z U = \frac{\partial U}{\partial z}. \tag{13.37b}$$

3. Gradient in Spherical Coordinates ($x = r \sin\vartheta \cos\varphi$, $y = r \sin\vartheta \sin\varphi$, $z = r \cos\vartheta$)

$$\operatorname{grad} U = \operatorname{grad}_r U \vec{\mathbf{e}}_r + \operatorname{grad}_\vartheta U \vec{\mathbf{e}}_\vartheta + \operatorname{grad}_\varphi U \vec{\mathbf{e}}_\varphi \quad \text{with} \tag{13.38a}$$

$$\operatorname{grad}_r U = \frac{\partial U}{\partial r}, \quad \operatorname{grad}_\vartheta U = \frac{1}{r}\frac{\partial U}{\partial \vartheta}, \quad \operatorname{grad}_\varphi U = \frac{1}{r \sin\vartheta}\frac{\partial U}{\partial \varphi}. \tag{13.38b}$$

4. Gradient in General Orthogonal Coordinates (ξ, η, ζ)

If $\vec{\mathbf{r}}(\xi,\eta,\zeta) = x(\xi,\eta,\zeta)\vec{\mathbf{i}} + y(\xi,\eta,\zeta)\vec{\mathbf{j}} + z(\xi,\eta,\zeta)\vec{\mathbf{k}}$, then we get

$$\operatorname{grad} U = \operatorname{grad}_\xi U \vec{\mathbf{e}}_\xi + \operatorname{grad}_\eta U \vec{\mathbf{e}}_\eta + \operatorname{grad}_\zeta U \vec{\mathbf{e}}_\zeta, \quad \text{where} \tag{13.39a}$$

$$\operatorname{grad}_\xi U = \frac{1}{\left|\frac{\partial \vec{\mathbf{r}}}{\partial \xi}\right|}\frac{\partial U}{\partial \xi}, \quad \operatorname{grad}_\eta U = \frac{1}{\left|\frac{\partial \vec{\mathbf{r}}}{\partial \eta}\right|}\frac{\partial U}{\partial \eta}, \quad \operatorname{grad}_\zeta U = \frac{1}{\left|\frac{\partial \vec{\mathbf{r}}}{\partial \zeta}\right|}\frac{\partial U}{\partial \zeta}. \tag{13.39b}$$

13.2.2.6 Rules of Calculations

We assume in the followings that $\vec{\mathbf{c}}$ and c are constant.

$$\operatorname{grad} c = \vec{\mathbf{0}}, \quad \operatorname{grad}(U_1 + U_2) = \operatorname{grad} U_1 + \operatorname{grad} U_2, \quad \operatorname{grad}(c\,U) = c \operatorname{grad} U. \tag{13.40}$$

$$\operatorname{grad}(U_1 U_2) = U_1 \operatorname{grad} U_2 + U_2 \operatorname{grad} U_1, \quad \operatorname{grad}\varphi(U) = \frac{d\varphi}{dU}\operatorname{grad} U. \tag{13.41}$$

$$\operatorname{grad}(\vec{\mathbf{V}}_1 \cdot \vec{\mathbf{V}}_2) = (\vec{\mathbf{V}}_1 \cdot \operatorname{grad})\vec{\mathbf{V}}_2 + (\vec{\mathbf{V}}_2 \cdot \operatorname{grad})\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_1 \times \operatorname{rot}\vec{\mathbf{V}}_2 + \vec{\mathbf{V}}_2 \times \operatorname{rot}\vec{\mathbf{V}}_1. \tag{13.42}$$

$$\operatorname{grad}(\vec{\mathbf{r}} \cdot \vec{\mathbf{c}}) = \vec{\mathbf{c}}. \tag{13.43}$$

1. Differential of a Scalar Field as the Total Differential of the Function U

$$dU = \operatorname{grad} U \cdot d\vec{\mathbf{r}} = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz. \tag{13.44}$$

2. Derivative of a Function U along a Space Curve $\vec{\mathbf{r}}(t)$

$$\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt}. \tag{13.45}$$

3. Gradient of a Central Field

$$\operatorname{grad} U(r) = U'(r)\frac{\vec{\mathbf{r}}}{r} \quad \text{(spherical field)}, \tag{13.46a}$$

$$\operatorname{grad} r = \frac{\vec{\mathbf{r}}}{r} \quad \text{(field of unit vectors)}. \tag{13.46b}$$

13.2.3 Vector Gradient

The relation (13.32a) inspires the notation

$$\frac{\partial \vec{\mathbf{V}}}{\partial \vec{\mathbf{a}}} = \vec{\mathbf{a}} \cdot \operatorname{grad}(V_x\vec{\mathbf{e}}_x + V_y\vec{\mathbf{e}}_y + V_z\vec{\mathbf{e}}_z) = \vec{\mathbf{a}} \cdot \operatorname{grad}\vec{\mathbf{V}} \tag{13.47a}$$

where $\operatorname{grad}\vec{\mathbf{V}}$ is called the *vector gradient*. It follows from the matrix notation of (13.47a) that the vector gradient, as a tensor, can be represented by a matrix:

$$(\vec{\mathbf{a}} \cdot \operatorname{grad})\vec{\mathbf{V}} = \begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} & \frac{\partial V_x}{\partial z} \\ \frac{\partial V_y}{\partial x} & \frac{\partial V_y}{\partial y} & \frac{\partial V_y}{\partial z} \\ \frac{\partial V_z}{\partial x} & \frac{\partial V_z}{\partial y} & \frac{\partial V_z}{\partial z} \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}, \tag{13.47b}$$

$$\operatorname{grad}\vec{\mathbf{V}} = \begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} & \frac{\partial V_x}{\partial z} \\ \frac{\partial V_y}{\partial x} & \frac{\partial V_y}{\partial y} & \frac{\partial V_y}{\partial z} \\ \frac{\partial V_z}{\partial x} & \frac{\partial V_z}{\partial y} & \frac{\partial V_z}{\partial z} \end{pmatrix}. \tag{13.47c}$$

These types of tensors have a very important role in engineering sciences, e.g., for the description of tension and elasticity (see 4.3.2, **4.**, p. 263, and p. 263).

13.2.4 Divergence of Vector Fields

13.2.4.1 Definition of Divergence

We can assign a scalar field to a vector field $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ which is called its *divergence*. The divergence is defined as a space derivative of the vector field at a point $\vec{\mathbf{r}}$:

$$\operatorname{div}\vec{\mathbf{V}} = \lim_{V\to 0}\frac{\displaystyle\oiint_{(\Sigma)}\vec{\mathbf{V}}\cdot d\vec{\mathbf{S}}}{V}. \tag{13.48}$$

If the vector field $\vec{\mathbf{V}}$ is considered as a stream field, then the divergence can be considered as the fluid output or source, because it gives the amount of fluid given in a unit of volume during a unit of time flowing by the considered point of the vector field $\vec{\mathbf{V}}$. In the case $\operatorname{div}\vec{\mathbf{V}} > 0$ the point is called a *source*, in the case $\operatorname{div}\vec{\mathbf{V}} < 0$ it is called a *sink*.

13.2.4.2 Divergence in Different Coordinates

1. Divergence in Cartesian Coordinates

$$\operatorname{div}\vec{\mathbf{V}} = \frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z} \quad (13.49\text{a}) \qquad \text{with} \quad \vec{\mathbf{V}}(x,y,z) = V_x\vec{\mathbf{i}}+V_y\vec{\mathbf{j}}+V_z\vec{\mathbf{k}}. \quad (13.49\text{b})$$

The scalar field $\operatorname{div}\vec{\mathbf{V}}$ can be represented as the dot product of the nabla operator and the vector $\vec{\mathbf{V}}$ as

$$\operatorname{div}\vec{\mathbf{V}} = \nabla\cdot\vec{\mathbf{V}} \tag{13.49c}$$

and it is translation and rotation invariant, i.e., scalar invariant (see 4.3.3.2, p. 265).

2. Divergence in Cylindrical Coordinates

$$\operatorname{div}\vec{\mathbf{V}} = \frac{1}{\rho}\frac{\partial(\rho V_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial V_\varphi}{\partial\varphi}+\frac{\partial V_z}{\partial z} \quad (13.50\text{a}) \quad \text{with} \quad \vec{\mathbf{V}}(\rho,\varphi,z) = V_\rho\vec{\mathbf{e}}_\rho+V_\varphi\vec{\mathbf{e}}_\varphi+V_z\vec{\mathbf{e}}_z. \quad (13.50\text{b})$$

3. Divergence in Spherical Coordinates

$$\operatorname{div}\vec{\mathbf{V}} = \frac{1}{r^2}\frac{\partial(r^2V_r)}{\partial r}+\frac{1}{r\sin\vartheta}\frac{\partial(\sin\vartheta V_\vartheta)}{\partial\vartheta}+\frac{1}{r\sin\vartheta}\frac{\partial V_\varphi}{\partial\varphi} \tag{13.51a}$$

$$\text{with} \quad \vec{\mathbf{V}}(r,\vartheta,\varphi) = V_r\vec{\mathbf{e}}_r+V_\vartheta\vec{\mathbf{e}}_\vartheta+V_\varphi\vec{\mathbf{e}}_\varphi. \tag{13.51b}$$

4. Divergence in General Orthogonal Coordinates

$$\operatorname{div}\vec{\mathbf{V}} = \frac{1}{\mathrm{D}}\left\{\frac{\partial}{\partial\xi}\left(\left|\frac{\partial\vec{\mathbf{r}}}{\partial\eta}\right|\left|\frac{\partial\vec{\mathbf{r}}}{\partial\zeta}\right|V_\xi\right)+\frac{\partial}{\partial\eta}\left(\left|\frac{\partial\vec{\mathbf{r}}}{\partial\zeta}\right|\left|\frac{\partial\vec{\mathbf{r}}}{\partial\xi}\right|V_\eta\right)+\frac{\partial}{\partial\zeta}\left(\left|\frac{\partial\vec{\mathbf{r}}}{\partial\xi}\right|\left|\frac{\partial\vec{\mathbf{r}}}{\partial\eta}\right|V_\zeta\right)\right\} \tag{13.52a}$$

$$\text{with} \quad \vec{\mathbf{r}}(\xi,\eta,\zeta) = x(\xi,\eta,\zeta)\vec{\mathbf{i}}+y(\xi,\eta,\zeta)\vec{\mathbf{j}}+z(\xi,\eta,\zeta)\vec{\mathbf{k}}, \tag{13.52b}$$

$$\mathrm{D} = \left|\left(\frac{\partial\vec{\mathbf{r}}}{\partial\xi}\frac{\partial\vec{\mathbf{r}}}{\partial\eta}\frac{\partial\vec{\mathbf{r}}}{\partial\zeta}\right)\right| = \left|\frac{\partial\vec{\mathbf{r}}}{\partial\xi}\right|\cdot\left|\frac{\partial\vec{\mathbf{r}}}{\partial\eta}\right|\cdot\left|\frac{\partial\vec{\mathbf{r}}}{\partial\zeta}\right|, \tag{13.52c}$$

$$\text{and} \quad \vec{\mathbf{V}}(\xi,\eta,\zeta) = V_\xi\vec{\mathbf{e}}_\xi+V_\eta\vec{\mathbf{e}}_\eta+V_\zeta\vec{\mathbf{e}}_\zeta. \tag{13.52d}$$

13.2.4.3 Rules for Evaluation of the Divergence

$$\operatorname{div}\vec{\mathbf{c}} = 0, \quad \operatorname{div}(\vec{\mathbf{V}}_1+\vec{\mathbf{V}}_2) = \operatorname{div}\vec{\mathbf{V}}_1+\operatorname{div}\vec{\mathbf{V}}_2, \quad \operatorname{div}(c\vec{\mathbf{V}}) = c\operatorname{div}\vec{\mathbf{V}}. \tag{13.53}$$

$$\operatorname{div}(U\vec{\mathbf{V}}) = U \operatorname{div}\vec{\mathbf{V}} + \vec{\mathbf{V}} \cdot \operatorname{grad} U \qquad \left(\text{especially } \operatorname{div}(r\vec{\mathbf{c}}) = \frac{\vec{\mathbf{r}} \cdot \vec{\mathbf{c}}}{r}\right). \tag{13.54}$$

$$\operatorname{div}(\vec{\mathbf{V}}_1 \times \vec{\mathbf{V}}_2) = \vec{\mathbf{V}}_2 \cdot \operatorname{rot}\vec{\mathbf{V}}_1 - \vec{\mathbf{V}}_1 \cdot \operatorname{rot}\vec{\mathbf{V}}_2 . \tag{13.55}$$

13.2.4.4 Divergence of a Central Field

$$\operatorname{div}\vec{\mathbf{r}} = 3, \quad \operatorname{div}\varphi(r)\vec{\mathbf{r}} = 3\varphi(r) + r\varphi'(r). \tag{13.56}$$

13.2.5 Rotation of Vector Fields

13.2.5.1 Definitions of the Rotation

1. Definition

The *rotation* or *curl* of a vector field $\vec{\mathbf{V}}$ at the point $\vec{\mathbf{r}}$ is a vector denoted by $\operatorname{rot}\vec{\mathbf{V}}$, $\operatorname{curl}\vec{\mathbf{V}}$ or with the nabla operator $\nabla \times \vec{\mathbf{V}}$, and defined as the negative space derivative of the vector field:

$$\operatorname{rot}\vec{\mathbf{V}} = -\lim_{V\to 0} \frac{\displaystyle\oiint_{(\Sigma)} \vec{\mathbf{V}} \times d\vec{\mathbf{S}}}{V} = \lim_{V\to 0} \frac{\displaystyle\oiint_{(\Sigma)} d\vec{\mathbf{S}} \times \vec{\mathbf{V}}}{V} . \tag{13.57}$$

2. Definition

The vector field of the rotation of the vector field $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ can be defined in the following way:

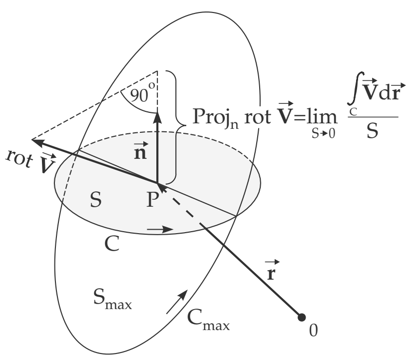

Figure 13.12

a) We put a small surface sheet S **(Fig. 13.12)** through the point $\vec{\mathbf{r}}$. We describe this surface sheet by a vector $\vec{\mathbf{S}}$ whose direction is the direction of the surface normal $\vec{\mathbf{n}}$ and its absolute value is equal to the area of this surface patch. The boundary of this surface is denoted by C.

b) We evaluate the integral $\displaystyle\oint_{(C)} \vec{\mathbf{V}} \cdot d\vec{\mathbf{r}}$ along the closed boundary curve C of the surface (the sense of the curve is positive looking to the surface from the direction of the surface normal (see **Fig. 13.12**).

c) We find the limit (if it exists) $\displaystyle\lim_{S\to 0} \frac{1}{S} \oint_{(C)} \vec{\mathbf{V}} \cdot d\vec{\mathbf{r}}$, while the position of the surface sheet remains unchanged.

d) We change the position of the surface sheet in order to get a maximum value of the limit. The surface area in this position is $S_{\max}$ and the corresponding boundary curve is $C_{\max}$.

e) We determine the vector $\operatorname{rot}\vec{\mathbf{r}}$ at the point $\vec{\mathbf{r}}$, whose absolute value is equal to the maximum value found above and its direction coincides with the direction of the surface normal of the corresponding surface. We then get:

$$\left|\operatorname{rot}\vec{\mathbf{V}}\right| = \lim_{S_{\max}\to 0} \frac{\displaystyle\oint_{(C_{\max})} \vec{\mathbf{V}} \cdot d\vec{\mathbf{r}}}{S_{\max}}. \tag{13.58a}$$

The projection of $\operatorname{rot}\vec{\mathbf{V}}$ onto the surface normal $\vec{\mathbf{n}}$ of a surface with area S, i.e., the component of the vector $\operatorname{rot}\vec{\mathbf{V}}$ in an arbitrary direction $\vec{\mathbf{n}} = \vec{\mathbf{l}}$ is

$$\vec{\mathbf{l}} \cdot \operatorname{rot} \vec{\mathbf{V}} = \operatorname{rot}_{l} \vec{\mathbf{V}} = \lim_{S \to 0} \frac{\oint\limits_{(C)} \vec{\mathbf{V}} \cdot d\vec{\mathbf{r}}}{S} . \tag{13.58b}$$

The vector lines of the field rot $\vec{\mathbf{V}}$ are called the *curl lines of the vector field* $\vec{\mathbf{V}}$.

13.2.5.2 Rotation in Different Coordinates

1. Rotation in Cartesian Coordinates

$$\operatorname{rot} \vec{\mathbf{V}} = \vec{\mathbf{i}} \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) + \vec{\mathbf{j}} \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) + \vec{\mathbf{k}} \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} . \tag{13.59a}$$

The vector field rot $\vec{\mathbf{V}}$ can be represented as the cross product of the nabla operator and the vector $\vec{\mathbf{V}}$:

$$\operatorname{rot} \vec{\mathbf{V}} = \nabla \times \vec{\mathbf{V}}. \tag{13.59b}$$

2. Rotation in Cylindrical Coordinates

$$\operatorname{rot} \vec{\mathbf{V}} = \operatorname{rot}_{\rho} \vec{\mathbf{V}} \vec{\mathbf{e}}_{\rho} + \operatorname{rot}_{\varphi} \vec{\mathbf{V}} \vec{\mathbf{e}}_{\varphi} + \operatorname{rot}_{z} \vec{\mathbf{V}} \vec{\mathbf{e}}_{z} \quad \text{with} \tag{13.60a}$$

$$\operatorname{rot}_{\rho} \vec{\mathbf{V}} = \frac{1}{\rho} \frac{\partial V_z}{\partial \varphi} - \frac{\partial V_{\varphi}}{\partial z}, \quad \operatorname{rot}_{\varphi} \vec{\mathbf{V}} = \frac{\partial V_{\rho}}{\partial z} - \frac{\partial V_z}{\partial \rho}, \quad \operatorname{rot}_{z} \vec{\mathbf{V}} = \frac{1}{\rho} \left\{ \frac{\partial}{\partial \rho} (\rho V_{\varphi}) - \frac{\partial V_{\rho}}{\partial \varphi} \right\} . \tag{13.60b}$$

3. Rotation in Spherical Coordinates

$$\operatorname{rot} \vec{\mathbf{V}} = \operatorname{rot}_{r} \vec{\mathbf{V}} \vec{\mathbf{e}}_{r} + \operatorname{rot}_{\vartheta} \vec{\mathbf{V}} \vec{\mathbf{e}}_{\vartheta} + \operatorname{rot}_{\varphi} \vec{\mathbf{V}} \vec{\mathbf{e}}_{\varphi} \quad \text{with} \tag{13.61a}$$

$$\left. \begin{aligned} \operatorname{rot}_{r} \vec{\mathbf{V}} &= \frac{1}{r \sin \vartheta} \left\{ \frac{\partial}{\partial \vartheta} (\sin \vartheta V_{\varphi}) - \frac{\partial V_{\vartheta}}{\partial \varphi} \right\}, \\ \operatorname{rot}_{\vartheta} \vec{\mathbf{V}} &= \frac{1}{r \sin \vartheta} \frac{\partial V_r}{\partial \varphi} - \frac{1}{r} \frac{\partial}{\partial r} (r V_{\varphi}), \\ \operatorname{rot}_{\varphi} \vec{\mathbf{V}} &= \frac{1}{r} \left\{ \frac{\partial}{\partial r} (r V_{\vartheta}) - \frac{V_r}{\partial \vartheta} \right\} . \end{aligned} \right\} \tag{13.61b}$$

4. Rotation in General Orthogonal Coordinates

$$\operatorname{rot} \vec{\mathbf{V}} = \operatorname{rot}_{\xi} \vec{\mathbf{V}} \vec{\mathbf{e}}_{\xi} + \operatorname{rot}_{\eta} \vec{\mathbf{V}} \vec{\mathbf{e}}_{\eta} + \operatorname{rot}_{\zeta} \vec{\mathbf{V}} \vec{\mathbf{e}}_{\zeta} \quad \text{with} \tag{13.62a}$$

$$\left. \begin{aligned} \operatorname{rot}_{\xi} \vec{\mathbf{V}} &= \frac{1}{\mathrm{D}} \left| \frac{\partial \vec{\mathbf{r}}}{\partial \xi} \right| \left[\frac{\partial}{\partial \eta} \left(\left| \frac{\partial \vec{\mathbf{r}}}{\partial \zeta} \right| V_{\zeta} \right) - \frac{\partial}{\partial \zeta} \left(\left| \frac{\partial \vec{\mathbf{r}}}{\partial \eta} \right| V_{\eta} \right) \right], \\ \operatorname{rot}_{\eta} \vec{\mathbf{V}} &= \frac{1}{\mathrm{D}} \left| \frac{\partial \vec{\mathbf{r}}}{\partial \eta} \right| \left[\frac{\partial}{\partial \zeta} \left(\left| \frac{\partial \vec{\mathbf{r}}}{\partial \xi} \right| V_{\xi} \right) - \frac{\partial}{\partial \xi} \left(\left| \frac{\partial \vec{\mathbf{r}}}{\partial \zeta} \right| V_{\zeta} \right) \right], \\ \operatorname{rot}_{\zeta} \vec{\mathbf{V}} &= \frac{1}{\mathrm{D}} \left| \frac{\partial \vec{\mathbf{r}}}{\partial \zeta} \right| \left[\frac{\partial}{\partial \xi} \left(\left| \frac{\partial \vec{\mathbf{r}}}{\partial \eta} \right| V_{\eta} \right) - \frac{\partial}{\partial \eta} \left(\left| \frac{\partial \vec{\mathbf{r}}}{\partial \xi} \right| V_{\xi} \right) \right], \end{aligned} \right\} \tag{13.62b}$$

$$\vec{\mathbf{r}}(\xi, \eta, \zeta) = x(\xi, \eta, \zeta) \vec{\mathbf{i}} + y(\xi, \eta, \zeta) \vec{\mathbf{j}} + z(\xi, \eta, \zeta) \vec{\mathbf{k}}; \qquad \mathrm{D} = \left| \frac{\partial \vec{\mathbf{r}}}{\partial \xi} \right| \cdot \left| \frac{\partial \vec{\mathbf{r}}}{\partial \eta} \right| \cdot \left| \frac{\partial \vec{\mathbf{r}}}{\partial \zeta} \right| . \tag{13.62c}$$

13.2.5.3 Rules for Evaluating the Rotation

$$\operatorname{rot} (\vec{\mathbf{V}}_{\mathbf{1}} + \vec{\mathbf{V}}_{\mathbf{2}}) = \operatorname{rot} \vec{\mathbf{V}}_{\mathbf{1}} + \operatorname{rot} \vec{\mathbf{V}}_{\mathbf{2}}, \quad \operatorname{rot} (c\vec{\mathbf{V}}) = c \operatorname{rot} \vec{\mathbf{V}}. \tag{13.63}$$

$$\operatorname{rot}(U\vec{\mathbf{V}}) = U \operatorname{rot}\vec{\mathbf{V}} + \operatorname{grad} U \times \vec{\mathbf{V}}. \tag{13.64}$$

$$\operatorname{rot}(\vec{\mathbf{V}}_1 \times \vec{\mathbf{V}}_2) = (\vec{\mathbf{V}}_2 \cdot \operatorname{grad})\vec{\mathbf{V}}_1 - (\vec{\mathbf{V}}_1 \cdot \operatorname{grad})\vec{\mathbf{V}}_2 + \vec{\mathbf{V}}_1 \operatorname{div}\vec{\mathbf{V}}_2 - \vec{\mathbf{V}}_2 \operatorname{div}\vec{\mathbf{V}}_1. \tag{13.65}$$

13.2.5.4 Rotation of a Potential Field

This also follows from the Stokes theorem (see 13.3.3.2, p. 666) that the rotation of a potential field is identically zero:

$$\operatorname{rot}\,\vec{\mathbf{V}} = \operatorname{rot}(\operatorname{grad}\,\mathrm{U}) = \vec{\mathbf{0}}. \tag{13.66}$$

This also follows from (13.59a) for $\vec{\mathbf{V}} = \operatorname{grad} U$, if the assumptions of the Schwarz interchanging theorem are fulfilled (see 6.2.2.2, **1.**, p. 395).

■ For $\vec{\mathbf{r}} = x\vec{\mathbf{i}} + y\vec{\mathbf{j}} + z\vec{\mathbf{k}}$ with $r = |\vec{\mathbf{r}}| = \sqrt{x^2+y^2+z^2}$ we have: $\operatorname{rot}\,\vec{\mathbf{r}} = 0$ and $\operatorname{rot}\,(\varphi(r)\vec{\mathbf{r}}) = \vec{\mathbf{0}}$, where $\varphi(r)$ is a differentiable function of r.

13.2.6 Nabla Operator, Laplace Operator

13.2.6.1 Nabla Operator

The symbolic vector ∇ is called the *nabla operator*. Its use simplifies the representation of and calculations with space differential operators. In Cartesian coordinates we have:

$$\nabla = \frac{\partial}{\partial x}\vec{\mathbf{i}} + \frac{\partial}{\partial y}\vec{\mathbf{j}} + \frac{\partial}{\partial z}\vec{\mathbf{k}}. \tag{13.67}$$

The components of the nabla operator are considered as partial differential operators, i.e., the symbol $\dfrac{\partial}{\partial x}$ means partial differentiation with respect to x, where the other variables are considered as constants.

The formulas for *spatial differential operators* in Cartesian coordinates can be obtained by formal multiplication of this vector operator by the scalar U or by the vector $\vec{\mathbf{V}}$. For instance, in the case of the operators *gradient, vector gradient, divergence*, and *rotation*:

$$\operatorname{grad} U = \nabla U \quad \text{(gradient of } U \text{)} \quad \text{(see 13.2.2, p. 650)),} \tag{13.68a}$$

$$\operatorname{grad}\vec{\mathbf{V}} = \nabla\vec{\mathbf{V}} \quad \text{(vector gradient of } \vec{\mathbf{V}} \text{)} \quad \text{(see 13.2.3, p. 652)),} \tag{13.68b}$$

$$\operatorname{div}\vec{\mathbf{V}} = \nabla\cdot\vec{\mathbf{V}} \quad \text{(divergence of } \vec{\mathbf{V}} \text{)} \quad \text{(see 13.2.4, p. 653)),} \tag{13.68c}$$

$$\operatorname{rot}\vec{\mathbf{V}} = \nabla\times\vec{\mathbf{V}} \quad \text{(rotation or curl of } \vec{\mathbf{V}} \text{)} \quad \text{(see 13.2.5, p. 654)).} \tag{13.68d}$$

13.2.6.2 Rules for Calculations with the Nabla Operator

1. If ∇ stands in front of a linear combination $\sum a_i X_i$ with constants a_i and with point functions X_i, then, independently of whether they are scalar or vector functions, we have the formula:

$$\nabla(\sum a_i X_i) = \sum a_i \nabla X_i. \tag{13.69}$$

2. If ∇ is applied to a product of scalar or vector functions, then we apply it to each of these functions after each other and add the result. There is a $\downarrow$ above the symbol of the function submitted to the operation

$$\nabla(XYZ) = \nabla(\overset{\downarrow}{X}YZ) + \nabla(X\overset{\downarrow}{Y}Z) + \nabla(XY\overset{\downarrow}{Z}), \quad \text{i.e.,} \tag{13.70}$$

$$\nabla(XYZ) = (\nabla X)YZ + X(\nabla Y)Z) + XY(\nabla Z).$$

We transform the products according to vector algebra so as the operator ∇ is applied to only one factor with the sign $\downarrow$. Having performed the computation we omit that sign.

■ **A:** $\operatorname{div}(U\vec{\mathbf{V}}) = \nabla(U\vec{\mathbf{V}}) = \nabla(\overset{\downarrow}{U}\,\vec{\mathbf{V}}) + \nabla(U\,\overset{\downarrow}{\vec{\mathbf{V}}}) = \vec{\mathbf{V}}\cdot\nabla U + U\nabla\cdot\vec{\mathbf{V}} = \vec{\mathbf{V}}\cdot\operatorname{grad}U + U\operatorname{div}\vec{\mathbf{V}}.$

■ **B:** $\operatorname{grad}(\vec{\mathbf{V}}_1\vec{\mathbf{V}}_2) = \nabla(\vec{\mathbf{V}}_1\vec{\mathbf{V}}_2) = \nabla(\overset{\downarrow}{\vec{\mathbf{V}}_1}\,\vec{\mathbf{V}}_2) + \nabla(\vec{\mathbf{V}}_1\,\overset{\downarrow}{\vec{\mathbf{V}}_2})$. Because $\vec{\mathbf{b}}(\vec{\mathbf{a}}\vec{\mathbf{c}}) = (\vec{\mathbf{a}}\vec{\mathbf{b}})\vec{\mathbf{c}} + \vec{\mathbf{a}}\times(\vec{\mathbf{b}}\times\vec{\mathbf{c}})$ we get: $\operatorname{grad}(\vec{\mathbf{V}}_1\vec{\mathbf{V}}_2) = (\vec{\mathbf{V}}_2\nabla)\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2\times(\nabla\times\vec{\mathbf{V}}_1) + (\vec{\mathbf{V}}_1\nabla)\vec{\mathbf{V}}_2 + \vec{\mathbf{V}}_1\times(\nabla\times\vec{\mathbf{V}}_2)$

$= (\vec{\mathbf{V}}_2\operatorname{grad})\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2\times\operatorname{rot}\vec{\mathbf{V}}_1 + (\vec{\mathbf{V}}_1\operatorname{grad})\vec{\mathbf{V}}_2 + \vec{\mathbf{V}}_1\times\operatorname{rot}\vec{\mathbf{V}}_2.$

13.2.6.3 Vector Gradient

The vector gradient $\operatorname{grad}\vec{\mathbf{V}}$ is represented by the nabla operator as

$$\operatorname{grad}\vec{\mathbf{V}} = \nabla\vec{\mathbf{V}}. \tag{13.71a}$$

We get for the expression occurring in the vector gradient $(\vec{\mathbf{a}}\cdot\nabla)\vec{\mathbf{V}}$:

$$2(\vec{\mathbf{a}}\cdot\nabla)\vec{\mathbf{V}} = \operatorname{rot}(\vec{\mathbf{V}}\times\vec{\mathbf{a}}) + \operatorname{grad}(\vec{\mathbf{a}}\vec{\mathbf{V}}) + \vec{\mathbf{a}}\operatorname{div}\vec{\mathbf{V}} - \vec{\mathbf{V}}\operatorname{div}\vec{\mathbf{a}} - \vec{\mathbf{a}}\times\operatorname{rot}\vec{\mathbf{V}} - \vec{\mathbf{V}}\times\operatorname{rot}\vec{\mathbf{a}}. \tag{13.71b}$$

In particular we get for $\vec{\mathbf{r}} = x\vec{\mathbf{i}} + y\vec{\mathbf{j}} + z\vec{\mathbf{k}}$:

$$(\vec{\mathbf{a}}\cdot\nabla)\vec{\mathbf{r}} = \vec{\mathbf{a}}. \tag{13.71c}$$

13.2.6.4 Nabla Operator Applied Twice

For every field $\vec{\mathbf{V}}$:

$$\nabla(\nabla\times\vec{\mathbf{V}}) = \operatorname{div}\operatorname{rot}\vec{\mathbf{V}} = 0, \tag{13.72a}$$

$$\nabla\times(\nabla U) = \operatorname{rot}\operatorname{grad}U = \vec{\mathbf{0}}, \tag{13.72b}$$

$$\nabla(\nabla U) = \operatorname{div}\operatorname{grad}U = \Delta U. \tag{13.72c}$$

13.2.6.5 Laplace Operator

1. Definition

The dot product of the nabla operator with itself is called the *Laplace operator*:

$$\Delta = \nabla\cdot\nabla = \nabla^2. \tag{13.73}$$

The Laplace operator is not a vector. It prescribes the summation of the second partial derivatives. It can be applied to scalar functions as well as to vector functions. The application to a vector function, componentwise, results in a vector.

The Laplace operator is an *invariant*, i.e., it does not change during translation and/or rotation of the coordinate system.

2. Formulas for the Laplace Operator in Different Coordinates

In the following, we apply the Laplace operator to the scalar point function $U(\vec{\mathbf{r}})$. Then the result is a scalar. The application of it for vector functions $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ results in a vector $\Delta\vec{\mathbf{V}}$ with components ΔV_x, ΔV_y, ΔV_z.

1. Laplace Operator in Cartesian Coordinates

$$\Delta U(x,y,z) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\,. \tag{13.74}$$

2. Laplace Operator in Cylindrical Coordinates

$$\Delta U(\rho,\varphi,z) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial U}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial\varphi^2} + \frac{\partial^2 U}{\partial z^2}\,. \tag{13.75}$$

3. Laplace Operator in Spherical Coordinates

$$\Delta U(r,\vartheta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial U}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 U}{\partial\varphi^2}\,. \tag{13.76}$$

4. Laplace Operator in General Orthogonal Coordinates

$$\Delta U(\xi,\eta,\zeta) = \frac{1}{\mathrm{D}}\left[\frac{\partial}{\partial\xi}\left(\frac{\mathrm{D}}{\left|\frac{\partial\vec{\mathbf{r}}}{\partial\xi}\right|^2}\frac{\partial U}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{\mathrm{D}}{\left|\frac{\partial\vec{\mathbf{r}}}{\partial\eta}\right|^2}\frac{\partial U}{\partial\eta}\right) + \frac{\partial}{\partial\zeta}\left(\frac{\mathrm{D}}{\left|\frac{\partial\vec{\mathbf{r}}}{\partial\zeta}\right|^2}\frac{\partial U}{\partial\zeta}\right)\right] \quad \text{with} \tag{13.77a}$$

$$\vec{\mathbf{r}}(\xi,\eta,\zeta) = x(\xi,\eta,\zeta)\vec{\mathbf{i}} + y(\xi,\eta,\zeta)\vec{\mathbf{j}} + z(\xi,\eta,\zeta)\vec{\mathbf{k}}, \quad (13.77\text{b}) \qquad \mathrm{D} = \left|\frac{\partial\vec{\mathbf{r}}}{\partial\xi}\right|\cdot\left|\frac{\partial\vec{\mathbf{r}}}{\partial\eta}\right|\cdot\left|\frac{\partial\vec{\mathbf{r}}}{\partial\zeta}\right|. \quad (13.77\text{c})$$

3. Special Relations between the Nabla Operator and Laplace Operator

$$\nabla(\nabla\cdot\vec{\mathbf{V}}) = \operatorname{grad}\operatorname{div}\vec{\mathbf{V}}, \tag{13.78}$$

$$\nabla\times(\nabla\times\vec{\mathbf{V}}) = \operatorname{rot}\operatorname{rot}\vec{\mathbf{V}}, \tag{13.79}$$

$$\nabla(\nabla\cdot\vec{\mathbf{V}}) - \nabla\times(\nabla\times\vec{\mathbf{V}}) = \Delta\vec{\mathbf{V}}, \quad \text{where} \tag{13.80}$$

$$\Delta\vec{\mathbf{V}} = (\nabla\cdot\nabla)\vec{\mathbf{V}} = \Delta V_x\vec{\mathbf{i}} + \Delta V_y\vec{\mathbf{j}} + \Delta V_z\vec{\mathbf{k}} = \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right)\vec{\mathbf{i}}$$
$$+\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right)\vec{\mathbf{j}} + \left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right)\vec{\mathbf{k}}. \tag{13.81}$$

13.2.7 Review of Spatial Differential Operations

13.2.7.1 Fundamental Relations and Results (see Table 13.2)

Table 13.2 Fundamental relations for spatial differential operators

Operator	Symbol	Relation	Argument	Result	Meaning
Gradient	grad U	∇U	scalar	vector	maximal increase
Vector gradient	grad $\vec{\mathbf{V}}$	$\nabla\vec{\mathbf{V}}$	vector	tensor second order	
Divergence	div $\vec{\mathbf{V}}$	$\nabla\cdot\vec{\mathbf{V}}$	vector	scalar	source, sink
Rotation	rot $\vec{\mathbf{V}}$	$\nabla\times\vec{\mathbf{V}}$	vector	vector	curl
Laplace operator	ΔU	$(\nabla\cdot\nabla)U$	scalar	scalar	potential field source
Laplace operator	$\Delta\vec{\mathbf{V}}$	$(\nabla\cdot\nabla)\vec{\mathbf{V}}$	vector	vector	

13.2.7.2 Rules of Calculation for Spatial Differential Operators

U, U_1, U_2, scalar functions; c constant; $\vec{\mathbf{V}}, \vec{\mathbf{V}}_1, \vec{\mathbf{V}}_2$ vector functions:

$$\operatorname{grad}(U_1 + U_2) = \operatorname{grad} U_1 + \operatorname{grad} U_2. \tag{13.82}$$

$$\operatorname{grad}(cU) = c\operatorname{grad} U. \tag{13.83}$$

$$\operatorname{grad}(U_1U_2) = U_1\operatorname{grad} U_2 + U_2\operatorname{grad} U_1. \tag{13.84}$$

$$\operatorname{grad} F(U) = F'(U)\operatorname{grad} U. \tag{13.85}$$

$$\operatorname{div}(\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2) = \operatorname{div}\vec{\mathbf{V}}_1 + \operatorname{div}\vec{\mathbf{V}}_2. \tag{13.86}$$

$$\operatorname{div}(c\vec{\mathbf{V}}) = c\operatorname{div}\vec{\mathbf{V}}. \tag{13.87}$$

$$\operatorname{div}(U\vec{\mathbf{V}}) = \vec{\mathbf{V}}\cdot\operatorname{grad}U + U\operatorname{div}\vec{\mathbf{V}}. \tag{13.88}$$

$$\operatorname{rot}(\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2) = \operatorname{rot}\vec{\mathbf{V}}_1 + \operatorname{rot}\vec{\mathbf{V}}_2. \tag{13.89}$$

$$\operatorname{rot}(c\vec{\mathbf{V}}) = c\operatorname{rot}\vec{\mathbf{V}}. \tag{13.90}$$

$$\operatorname{rot}(U\vec{\mathbf{V}}) = U\operatorname{rot}\vec{\mathbf{V}} - \vec{\mathbf{V}}\times\operatorname{grad}U. \tag{13.91}$$

$$\operatorname{div}\operatorname{rot}\vec{\mathbf{V}} \equiv 0. \tag{13.92}$$

$$\operatorname{rot}\operatorname{grad}U \equiv \vec{\mathbf{0}} \qquad \text{(zero vector)}. \tag{13.93}$$

$$\operatorname{div}\operatorname{grad}U = \Delta U. \tag{13.94}$$

$$\operatorname{rot}\operatorname{rot}\vec{\mathbf{V}} = \operatorname{grad}\operatorname{div}\vec{\mathbf{V}} - \Delta\vec{\mathbf{V}}. \tag{13.95}$$

$$\operatorname{div}(\vec{\mathbf{V}}_1 \times \vec{\mathbf{V}}_2) = \vec{\mathbf{V}}_2\cdot\operatorname{rot}\vec{\mathbf{V}}_1 - \vec{\mathbf{V}}_1\cdot\operatorname{rot}\vec{\mathbf{V}}_2. \tag{13.96}$$

13.2.7.3 Expressions of Vector Analysis in Cartesian, Cylindrical, and Spherical Coordinates (see Table 13.3)

Table 13.3 Expressions of vector analysis in Cartesian, cylindrical, and spherical coordinates

	Cartesian coordinates	**Cylindrical coordinates**	**Spherical coordinates**
$d\vec{\mathbf{s}} = d\vec{\mathbf{r}}$	$\vec{\mathbf{e}}_x dx + \vec{\mathbf{e}}_y dy + \vec{\mathbf{e}}_z dz$	$\vec{\mathbf{e}}_\rho d\rho + \vec{\mathbf{e}}_\varphi \rho d\varphi + \vec{\mathbf{e}}_z dz$	$\vec{\mathbf{e}}_r dr + \vec{\mathbf{e}}_\vartheta r\,d\vartheta + \vec{\mathbf{e}}_\varphi r\sin\vartheta\, d\varphi$
grad$U$	$\vec{\mathbf{e}}_x\frac{\partial U}{\partial x} + \vec{\mathbf{e}}_y\frac{\partial U}{\partial y} + \vec{\mathbf{e}}_z\frac{\partial U}{\partial z}$	$\vec{\mathbf{e}}_\rho\frac{\partial U}{\partial \rho} + \vec{\mathbf{e}}_\varphi\frac{1}{\rho}\frac{\partial U}{\partial \varphi} + \vec{\mathbf{e}}_z\frac{\partial U}{\partial z}$	$\vec{\mathbf{e}}_r\frac{\partial U}{\partial r} + \vec{\mathbf{e}}_\vartheta\frac{1}{r}\frac{\partial U}{\partial \vartheta} + \vec{\mathbf{e}}_\varphi\frac{1}{r\sin\vartheta}\frac{\partial U}{\partial \varphi}$
div$\vec{\mathbf{V}}$	$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V_\varphi}{\partial \varphi} + \frac{\partial V_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \vartheta}(V_\vartheta\sin\vartheta)$ $+\frac{1}{r\sin\vartheta}\frac{\partial V_\varphi}{\partial \varphi}$
rot$\vec{\mathbf{V}}$	$\vec{\mathbf{e}}_x\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)$ $+\vec{\mathbf{e}}_y\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)$ $+\vec{\mathbf{e}}_z\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$	$\vec{\mathbf{e}}_\rho\left(\frac{1}{\rho}\frac{\partial V_z}{\partial \varphi} - \frac{\partial V_\varphi}{\partial z}\right)$ $+\vec{\mathbf{e}}_\varphi\left(\frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial \rho}\right)$ $+\vec{\mathbf{e}}_z\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho V_\varphi) - \frac{1}{\rho}\frac{\partial V_\rho}{\partial \varphi}\right)$	$\vec{\mathbf{e}}_r\frac{1}{r\sin\vartheta}\left[\frac{\partial}{\partial \vartheta}(V_\varphi\sin\vartheta) - \frac{\partial V_\vartheta}{\partial \varphi}\right]$ $+\vec{\mathbf{e}}_\vartheta\frac{1}{r}\left[\frac{1}{\sin\vartheta}\frac{\partial V_r}{\partial \varphi} - \frac{\partial}{\partial r}(rV_\varphi)\right]$ $+\vec{\mathbf{e}}_\varphi\frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\vartheta) - \frac{\partial V_r}{\partial \vartheta}\right]$
ΔU	$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \varphi^2}$ $+\frac{\partial^2 U}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)$ $+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta\frac{\partial U}{\partial \vartheta}\right)$ $+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 U}{\partial \varphi^2}$

13.3 Integration in Vector Fields

Integration in vector fields is usually performed in Cartesian, cylindrical or in spherical coordinate systems. Usually we integrate along curves, surfaces, or volumes. The line, surface, and volume elements needed for these calculations are collected in **Table 13.4**.

Table 13.4 Line, surface, and volume elements in Cartesian, cylindrical, and spherical coordinates

	Cartesian coordinates	**Cylindrical coordinates**	**Spherical coordinates**
$d\vec{\mathbf{r}}$	$\vec{\mathbf{e}}_x dx + \vec{\mathbf{e}}_y dy + \vec{\mathbf{e}}_z dz$	$\vec{\mathbf{e}}_\rho d\rho + \vec{\mathbf{e}}_\varphi \rho d\varphi + \vec{\mathbf{e}}_z dz$	$\vec{\mathbf{e}}_r dr + \vec{\mathbf{e}}_\vartheta r d\vartheta + \vec{\mathbf{e}}_\varphi r \sin\vartheta d\varphi$
$d\vec{\mathbf{S}}$	$\vec{\mathbf{e}}_x dy dz + \vec{\mathbf{e}}_y dx dz + \vec{\mathbf{e}}_z dx dy$	$\vec{\mathbf{e}}_\rho \rho d\varphi dz + \vec{\mathbf{e}}_\varphi d\rho dz + \vec{\mathbf{e}}_z \rho d\rho d\varphi$	$\vec{\mathbf{e}}_r r^2 \sin\vartheta d\vartheta d\varphi$ $+\vec{\mathbf{e}}_\vartheta r \sin\vartheta dr d\varphi$ $+\vec{\mathbf{e}}_\varphi r dr d\vartheta$
dv^*	$dx dy dz$	$\rho d\rho d\varphi dz$	$r^2 \sin\vartheta dr d\vartheta d\varphi$
	$\vec{\mathbf{e}}_x = \vec{\mathbf{e}}_y \times \vec{\mathbf{e}}_z$ $\vec{\mathbf{e}}_y = \vec{\mathbf{e}}_z \times \vec{\mathbf{e}}_x$ $\vec{\mathbf{e}}_z = \vec{\mathbf{e}}_x \times \vec{\mathbf{e}}_y$	$\vec{\mathbf{e}}_\rho = \vec{\mathbf{e}}_\varphi \times \vec{\mathbf{e}}_z$ $\vec{\mathbf{e}}_\varphi = \vec{\mathbf{e}}_z \times \vec{\mathbf{e}}_\rho$ $\vec{\mathbf{e}}_z = \vec{\mathbf{e}}_\rho \times \vec{\mathbf{e}}_\varphi$	$\vec{\mathbf{e}}_r = \vec{\mathbf{e}}_\vartheta \times \vec{\mathbf{e}}_\varphi$ $\vec{\mathbf{e}}_\vartheta = \vec{\mathbf{e}}_\varphi \times \vec{\mathbf{e}}_r$ $\vec{\mathbf{e}}_\varphi = \vec{\mathbf{e}}_r \times \vec{\mathbf{e}}_\vartheta$
	$\vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$	$\vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$	$\vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$
	The indices i and j take the place of x, y, z or ρ, φ, z or r, ϑ, φ.		
*	The volume is denoted here by v to avoid confusion with the absolute value of the vector function $\lvert\vec{\mathbf{V}}\rvert = V$.		

13.3.1 Line Integral and Potential in Vector Fields

13.3.1.1 Line Integral in Vector Fields

1. Definition The scalar-valued curvilinear integral or line integral of a vector function $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ along a rectifiable curve $\overset{\frown}{AB}$ **(Fig. 13.13)** is the scalar value

$$P = \int\limits_{\overset{\frown}{AB}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}}. \tag{13.97a}$$

2. Evaluation of this Integral in Five Steps

a) We divide the path $\overset{\frown}{AB}$ **(Fig. 13.13)** by division points $A_1(\vec{\mathbf{r}}_1), A_2(\vec{\mathbf{r}}_2), \ldots, A_{n-1}(\vec{\mathbf{r}}_{n-1})$ $(A \equiv A_0,$ $B \equiv A_n)$ into n small arcs which are approximated by the vectors $\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_{i-1} = \Delta\vec{\mathbf{r}}_{i-1}$.

b) We choose arbitrarily the points P_i with position vectors $\vec{\mathbf{r}}_i$ lying inside or at the boundary of each small arc.

c) We calculate the dot product of the value of the function $\vec{\mathbf{V}}(\vec{\mathbf{r}}_i)$ at these chosen points with the corresponding $\Delta\vec{\mathbf{r}}_{i-1}$.

d) We add all the n products.

e) We calculate the limit of the sums got this way $\sum\limits_{i=1}^{n} \tilde{\vec{\mathbf{V}}}(\vec{\mathbf{r}}_i) \cdot \Delta\vec{\mathbf{r}}_{i-1}$ for $\Delta\vec{\mathbf{r}}_{i-1} \to 0$, while $n \to \infty$ obviously.

If this limit exists independently of the choice of the points A_i and P_i, then it is called the line integral

$$\int\limits_{\widehat{AB}} \vec{\mathbf{V}} \cdot d\vec{\mathbf{r}} = \lim_{\substack{\Delta\vec{\mathbf{r}}\to 0 \\ n\to\infty}} \sum_{i=1}^{n} \tilde{\vec{\mathbf{V}}}(\vec{\mathbf{r}}_i) \cdot \Delta\vec{\mathbf{r}}_{i-1}. \tag{13.97b}$$

A sufficient condition for the existence of the line integral (13.97a,b) is that the vector function $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ and the curve $\widehat{AB}$ are continuous and the curve has a tangent varying continuously. A vector function $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ is continuous if its components, the three scalar functions, are continuous.

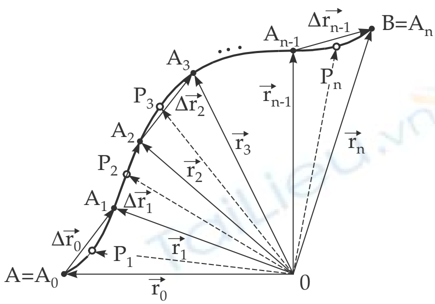

Figure 13.13

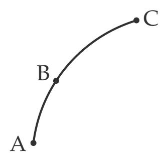

Figure 13.14

13.3.1.2 Interpretation of the Line Integral in Mechanics

If $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ is a field of force, i.e., $\vec{\mathbf{V}}(\vec{\mathbf{r}}) = \vec{\mathbf{F}}(\vec{\mathbf{r}})$, then the line integral (13.97a) represents the work done by $\vec{\mathbf{F}}$ while a particle m moves along the path $\widehat{AB}$ **(Fig. 13.13,13.14)**.

13.3.1.3 Properties of the Line Integral

$$\int\limits_{\widehat{ABC}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} = \int\limits_{\widehat{AB}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} + \int\limits_{\widehat{BC}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}}. \tag{13.98}$$

$$\int\limits_{\widehat{AB}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} = -\int\limits_{\widehat{BA}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} \qquad \textbf{(Fig. 13.14)}. \tag{13.99}$$

$$\int\limits_{\widehat{AB}} \left[\vec{\mathbf{V}}(\vec{\mathbf{r}}) + \vec{\mathbf{W}}(\vec{\mathbf{r}})\right] \cdot d\vec{\mathbf{r}} = \int\limits_{\widehat{AB}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} + \int\limits_{\widehat{AB}} \vec{\mathbf{W}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}}. \tag{13.100}$$

$$\int\limits_{\widehat{AB}} c\vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} = c\int\limits_{\widehat{AB}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}}. \tag{13.101}$$

13.3.1.4 Line Integral in Cartesian Coordinates

In Cartesian coordinates, we have:

$$\int\limits_{\widehat{AB}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} = \int\limits_{\widehat{AB}} (V_x\,dx + V_y\,dy + V_z\,dz)\,. \tag{13.102}$$