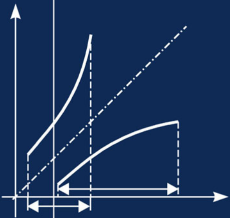

I. N. Bronshtein K. A. Semendyayev
G. Musiol H. Muehlig

HANDBOOK OF MATHEMATICS

Fifth Edition

Springer

I.N. Bronshtein · K.A. Semendyayev · G. Musiol · H. Muehlig

Handbook of Mathematics

I.N. Bronshtein · K.A. Semendyayev · G. Musiol · H. Muehlig

Handbook of Mathematics

5th Ed.

With 745 Figures and 142 Tables

Springer

Ilja N. Bronshtein †
Konstantin A. Semendyayev †
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Preface to the Fifth English Edition

This fifth edition is based on the fourth English edition (2003) and corresponds to the improved sixth German edition (2005). It contains all the chapters of the both mentioned editions, but in a renewed revised and extendet form.

So in the work at hand, the classical areas of Engineering Mathematics required for current practice are presented, such as “Arithmetic”, “Functions”, “Geometry”, “Linear Algebra”, “Algebra and Discrete Mathematics”, (including “Logic”, “Set Theory”, “Classical Algebraic Structures”, “Finite Fields”, “Elementary Number Theory”, ”Cryptology”, “Universal Algebra”, “Boolean Algebra and Swich Algebra”, “Algorithms of Graph Theory”, “Fuzzy Logic”), “Differentiation”, “Integral Calculus”, “Differential Equations”, “Calculus of Variations”, “Linear Integral Equations”, “Functional Analysis”, “Vector Analysis and Vector Fields”, “Function Theory”, “Integral Transformations”, “Probability Theory and Mathematical Statistics”.

Fields of mathematics that have gained importance with regards to the increasing mathematical modeling and penetration of technical and scientific processes also receive special attention. Included amongst these chapters are “Stochastic Processes and Stochastic Chains” as well as “Calculus of Errors”, “Dynamical Systems and Chaos”, “Optimization”, “Numerical Analysis”, “Using the Computer” and “Computer Algebra Systems”.

The chapter 21 containing a large number of useful tables for practical work has been completed by adding tables with the physical units of the International System of Units (SI).

Dresden, February 2007

Prof. Dr. Gerhard Musiol — Prof. Dr. Heiner Mühlig

From the Preface to the Fourth English Edition

The “Handbook of Mathematics” by the mathematician, I. N. Bronshtein and the engineer, K. A. Semendyayev was designed for engineers and students of technical universities. It appeared for the first time in Russian and was widely distributed both as a reference book and as a text book for colleges and universities. It was later translated into German and the many editions have made it a permanent fixture in German-speaking countries, where generations of engineers, natural scientists and others in technical training or already working with applications of mathematics have used it.

On behalf of the publishing house Harri Deutsch, a revision and a substantially enlarged edition was prepared in 1992 by Gerhard Musiol and Heiner Mühlig, with the goal of giving ”Bronshtein” the modern practical coverage requested by numerous students, university teachers and practitioners. The original style successfully used by the authors has been maintained. It can be characterized as “short, easily understandable, comfortable to use, but featuring mathematical accuracy (at a level of detail consistent with the needs of engineers)”*. Since 2000, the revised and extended fifth German edition of the revision has been on the market. Acknowledging the success that “Bronstein” has experienced in the German-speaking countries, Springer-Verlag Heidelberg/Germany is publishing a fourth English edition, which corresponds to the improved and extended fifth German edition.

The book is enhanced with over a thousand complementary illustrations and many tables. Special functions, series expansions, indefinite, definite and elliptic integrals as well as integral transformations

*See Preface to the First Russian Edition and statistical distributions are supplied in an extensive appendix of tables.

In order to make the reference book more effective, clarity and fast access through a clear structure were the goals, especially through visual clues as well as by a detailed technical index and colored tabs.

An extended bibliography also directs users to further resources.

We would like to cordially thank all readers and professional colleagues who helped us with their valuable statements, remarks and suggestions on the German edition of the book during the revision process. Special thanks go to Mrs. Professor Dr. Gabriela Szép (Budapest), who made this English debut version possible. Furthermore our thanks go to the co-authors for the critical treatment of their chapters.

Dresden, June 2003

Prof. Dr. GERHARD MUSIOL　　　　Prof. Dr. HEINER MÜHLIG

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